



Aggregation of triangle of distortion functions

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ABSTRACT

In this paper we introduce notions of distortion function and aggregated distortion function. Applying some extended aggregation function on the triangle of distortion functions, a new extended aggregation function is obtained. Properties of the aggregation function constructed in this way depend on the properties of applied aggregation function and the distortion functions used. Its properties as continuity, convexity, concavity, subadditivity and superadditivity are investigated.

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1. Introduction

This paper is a continuation of the earlier research on construction of new aggregation functions with desirable properties [1–10]. The main topic is a construction of new extended aggregation function by applying an initial extended aggregation function on the triangle of initial functions. In [1–4,7–9], an initial extended aggregation function is applied on the triangle of distance functions, see [11–14]. In this way, a new distance function is constructed and applied in image segmentation algorithm. We remark that in [6] a new fuzzy measure is constructed by aggregating a triangle of fuzzy measures, see [15–24], where the properties of the constructed fuzzy measure depend on the properties of the initial fuzzy measures and the properties of the applied extended aggregation function. In [10], an initial aggregation function is applied on the sequence of fuzzy metrics, see [25], where a new fuzzy metric is constructed and used for image segmentation and image denoising.

In this paper we introduce a notion of distortion functions which can be observed as a transformation of some measure-like information. Further, we will investigate the relevant properties of distortion functions such as continuity, convexity, concavity, subadditivity, superadditivity and some other properties which are important for their aggregation and

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construction of new distance functions, fuzzy measures and fuzzy metrics. The properties of convexity, concavity, subadditivity and superadditivity of distortion functions and aggregation functions are closely connected with inequalities of Jensen, Minkowski, Hölder etc., see [26–28]. Applying an initial extended aggregation function on a triangle of distortion functions, the new extended aggregation function is obtained. The properties of extended aggregation function constructed in this way depend on the properties of applied extended aggregation function and used distortion functions.

In Section 2, the notion of aggregation functions on bounded and unbounded intervals is presented, see [29–36]. Some additional properties of aggregation functions are listed in the same section. In Section 3 we introduce a notion of distortion function. In the same section, we proved some theorems related to continuity, convexity, concavity, subadditivity and superadditivity of distortion functions. In Section 4 we introduce a method of constructing new extended aggregation function by applying an initial extended aggregation function on the triangle of distortion functions. Some basic properties of the newly constructed aggregation function are discussed. In Section 5 we discuss subadditivity and superadditivity of the constructed extended aggregation function presented in Section 4. Here is proved under which additional conditions concerning properties of initial distortion functions and initial extended aggregation function the constructed function is subadditive or superadditive, with special accent on continuity, convexity and concavity. In Section 6 we analyze some other properties of the extended aggregation function presented in Section 4, which are relevant from the aspect of construction of fuzzy measures and distance functions. The above mentioned results and discussions are illustrated with several examples in Section 7. In Section 8 some conclusions are formulated for the research and results obtained in this paper.

2. Preliminaries and previous research

Aggregation functions are a wide class of fuzzy operators, see [29–34,36–41], and they are used as a tool for information fusion and decision making, in many scientific fields, especially in informatics, engineering, and even in social sciences, see [12,41–45].

For the purpose of various fields of application, it is often of interest to consider arguments and values in some other interval than $[0, 1]$ bounded and unbounded intervals of real numbers. For the purpose of the research carried out in this article, as in [6], we consider the following intervals:

[I1] Bounded interval $I_1 = [0, 1]$.

[I2] Unbounded real interval $I_2 = [0, \infty[$.

[I3] Unbounded real interval $I_3 = [0, \infty]$, where by convention we adopt that $\forall a \in [0, \infty]$, $a + \infty = \infty$, $0 \cdot \infty = 0$, $\infty - \infty = 0$ and $\forall a \in]0, \infty]$, $a \cdot \infty = \infty$.

We have by [30,31] the following definitions, where I is one of the considered intervals I_1, I_2 or I_3 .

Definition 1 (Aggregation function). For $n \geq 2$, an n -ary aggregation function is a function $A_{[n]} : I^n \rightarrow I$ with the following properties:

(a01) The following *boundary conditions* hold.

(B0)

$$A_{[n]}(0, \dots, 0) = 0.$$

(B1)

Depending on the observed interval I ,

[I1] for $I_1 = [0, 1]$,

$$A_{[n]}(1, \dots, 1) = 1,$$

[I2] for $I_2 = [0, \infty[$,

$$\lim_{\forall i \in \{1, \dots, n\}, a_i \rightarrow \infty} A_{[n]}(a_1, \dots, a_n) = \infty,$$

[I3] for $I_3 = [0, \infty]$,

$$A_{[n]}(\infty, \dots, \infty) = \infty.$$

(a02) A function A is monotonically non-decreasing in each component, i.e.,

$$\forall i \in \{1, \dots, n\}, a_i \leq b_i \Rightarrow A_{[n]}(a_1, \dots, a_n) \leq A_{[n]}(b_1, \dots, b_n) \quad (1)$$

hold for all n -tuples $(a_1, \dots, a_n), (b_1, \dots, b_n) \in I^n$.

By definition, for $n = 1$ is $A_{[1]}(x) = x, x \in I$.

Definition 2 (Extended aggregation function). An extended aggregation function is a function $A : \bigcup_{n=1}^{\infty} I^n \rightarrow I$ such that its every restriction $A_{[n]} : I^n \rightarrow I, n \in \mathbb{N}$ is an n -ary aggregation function.

Aggregation function can have many additional interesting and important properties, see [31].

Definition 3. For any fixed $n \in \mathbb{N}$, n -ary aggregation $A_{[n]}$ function may have the following properties:

(a03) Aggregation function $A_{[n]}$ is *continuous*.

(a04) Aggregation function $A_{[n]}$ is *symmetric in each component* if for each n -tuple $(a_1, \dots, a_n) \in I^n$ and each permutation p of the set $\{1, \dots, n\}$ holds that

$$A_{[n]}(a_1, \dots, a_n) = A_{[n]}(a_{p(1)}, \dots, a_{p(n)}).$$

(a05) Aggregation function $A_{[n]}$ is *idempotent* if for each n -tuple $(a, \dots, a) \in I^n$ holds that $A_{[n]}(a, \dots, a) = a$.

Remark 1. In the case of the interval $I_3 = [0, \infty]$ and points $(a_1, \dots, a_n) \in I_3^n$ which possible include also $a_i = \infty$ for some $i \in \{1, \dots, n\}$, continuity (a03) of n -ary aggregation function $A_{[n]}$ is understood in the sense that $\lim_{x_i \rightarrow a_i} A_{[n]}(x_1, \dots, x_n) = A_{[n]}(a_1, \dots, a_n)$ for all $(a_1, \dots, a_n) \in I_3^n$.

In [8] we have introduced a method of construction of new distance functions and metrics by applying an extended aggregation function on a given triangle of initial distance functions and metrics. An application of such constructed distance functions in image segmentation is presented in [1,3,7–9]. Similarly, in [6], for all considered intervals I_1, I_2 and I_3 , a general construction method for fuzzy measures by applying extended aggregation function on a given triangle of fuzzy measures is presented. The properties of the constructed distance functions and fuzzy measures depend both on the properties of the initial distance functions and fuzzy measures and the properties of the applied aggregation function, see [6,8]. Therefore the following properties of aggregation function are important, and will be considered in Section 4.

Definition 4. For any fixed $n \in \mathbb{N}$, let $A_{[n]}$ be an n -ary aggregation function.

(a06) Function $A_{[n]}$ is *additive* if for all $(a_1, \dots, a_n) \in I^n, (b_1, \dots, b_n) \in I^n$ that satisfy $(a_1 + b_1, \dots, a_n + b_n) \in I^n$ holds

$$A_{[n]}(a_1 + b_1, \dots, a_n + b_n) = A_{[n]}(a_1, \dots, a_n) + A_{[n]}(b_1, \dots, b_n).$$

(a07) Function $A_{[n]}$ is *subadditive* if for all $(a_1, \dots, a_n) \in I^n, (b_1, \dots, b_n) \in I^n$ that satisfy condition $(a_1 + b_1, \dots, a_n + b_n) \in I^n$ holds

$$A_{[n]}(a_1 + b_1, \dots, a_n + b_n) \leq A_{[n]}(a_1, \dots, a_n) + A_{[n]}(b_1, \dots, b_n).$$

(a08) Function $A_{[n]}$ is *superadditive* if for all $(a_1, \dots, a_n) \in I^n, (b_1, \dots, b_n) \in I^n$ that satisfy condition $(a_1 + b_1, \dots, a_n + b_n) \in I^n$ holds

$$A_{[n]}(a_1 + b_1, \dots, a_n + b_n) \geq A_{[n]}(a_1, \dots, a_n) + A_{[n]}(b_1, \dots, b_n).$$

(a09) Function $A_{[n]}$ is *positively homogenous* if for all $(a_1, \dots, a_n) \in I^n$ and for all $t \geq 0$ such that $(ta_1, \dots, ta_n) \in I^n$ holds $A_{[n]}(ta_1, \dots, ta_n) = tA_{[n]}(a_1, \dots, a_n)$.

(a10) Function $A_{[n]}$ is *positively subhomogenous* if for all $(a_1, \dots, a_n) \in I^n$ and $t \geq 0$ such that $(ta_1, \dots, ta_n) \in I^n$ holds $A_{[n]}(ta_1, \dots, ta_n) \leq tA_{[n]}(a_1, \dots, a_n)$.

(a11) Function $A_{[n]}$ is *positively superhomogenous* if for all $(a_1, \dots, a_n) \in I^n$ and $t \geq 0$ such that $(ta_1, \dots, ta_n) \in I^n$ holds $A_{[n]}(ta_1, \dots, ta_n) \geq tA_{[n]}(a_1, \dots, a_n)$.

(a12) Aggregation function $A_{[n]}$ satisfy the implication

$$A_{[n]}(a_1, \dots, a_n) = 0 \Rightarrow \forall i \in \{1, \dots, n\}, a_i = 0.$$

(a13) Aggregation function $A_{[n]}$ satisfy the implication

$$A_{[n]}(a_1, \dots, a_n) = 0 \Rightarrow \exists i \in \{1, \dots, n\}, a_i = 0.$$

(a14) Aggregation function $A_{[n]}$, depending on considered interval I satisfy one of the following implications.

[I1] For $I_1 = [0, 1]$,

$$A_{[n]}(a_1, \dots, a_n) < 1 \Rightarrow \forall i \in \{1, \dots, n\}, a_i < 1.$$

- [I2] For $I_2 = [0, \infty[$ and $(b_1, \dots, b_n) \in [0, \infty]^n$,
 $\lim_{\substack{a_i \in \{1, \dots, n\}, \\ a_i \rightarrow b_i}} A_{[n]}(a_1, \dots, a_n) < \infty \Rightarrow \forall i \in \{1, \dots, n\}, b_i < \infty.$
- [I3] For $I_3 = [0, \infty]$,
 $A_{[n]}(a_1, \dots, a_n) < \infty \Rightarrow \forall i \in \{1, \dots, n\}, a_i < \infty.$

Remark 2. Aggregation function does not necessarily satisfy the property (a14) even in the case of interval I_2 . For example, for $n = 2$, consider the well known aggregation function minimum (see [31]) $A_{[2]} : [0, \infty]^2 \rightarrow [0, \infty[$ defined by $A_{[2]}(a_1, a_2) = \min(a_1, a_2)$, $a_1, a_2 \in [0, \infty[$. Aggregation function $A_{[2]}$ does not satisfy the property (a14) since, e.g., $\lim_{(a_1, a_2) \rightarrow (5, \infty)} \min(a_1, a_2) = 5 < \infty$.

For extended aggregation function $A : \bigcup_{n=1}^{\infty} I^n \rightarrow I$, we will say that A has some of the considered property if its every restriction $A_{[n]} : I^n \rightarrow I$, $n \in \mathbb{N}$ has that property.

One widely used class of aggregation functions is *weighted averages*, see [29,31]. Each aggregation function of this type is associated with appropriated triangle of “weights”.

Definition 5. For each $n \in \mathbb{N}$, let $\omega_{[n]} = (\omega_{n,1}, \dots, \omega_{n,n})$ be an n -tuple of nonnegative real numbers $\omega_{n,i} \geq 0$, $i \in \{1, \dots, n\}$, such that

- $\sum_{i=1}^n \omega_{n,i} = 1$ in the case of interval $I_1 = [0, 1]$,
- $\omega_{n,i_0} > 0$ for some $i_0 \in \{1, \dots, n\}$ in the cases of intervals $I_2 = [0, \infty[$ and $I_3 = [0, \infty]$.

System $\omega = \{\omega_{[n]} \mid n \in \mathbb{N}\}$ of such coefficients is called *triangle of weights*.

One simple and well known extended aggregation function is described in the following example, see [1,6,29,31,36].

Example 1. Let $\omega = \{\omega_{[n]} \mid n \in \mathbb{N}\}$ be a triangle of weights. Function $\text{WAM}_{[n]}^{\omega_{[n]}} : I^n \rightarrow I$ defined with

$$\text{WAM}_{[n]}^{\omega_{[n]}}(a_1, \dots, a_n) = \sum_{i=1}^n \omega_{n,i} a_i, \quad (a_1, \dots, a_n) \in I^n \quad (2)$$

is called *weighted arithmetic mean*. In the case when $\omega_{n,i} = \frac{1}{n}$, $i \in \{1, \dots, n\}$, aggregation function $\text{WAM}_{[n]}^{\omega_{[n]}}$ reduces to *arithmetic mean*. It is easy to verify which of the properties from Definitions 3 and 4 are satisfied for each n -ary aggregation function $\text{WAM}_{[n]}^{\omega_{[n]}}$ and the corresponding extended aggregation function WAM^{ω} , see [1,6]. ✓

Regarding the possible additional properties of aggregation function listed in Definitions 3 and 4, some remarks follow.

Remark 3.

- Aggregation function $A_{[1]}(x)$, $x \in I$ obviously satisfies all the properties from Definitions 3 and 4, and because of that, further considerations and examinations of the properties of aggregation functions $A_{[n]}$ will be carried for $n \geq 2$.
- Assumptions $(a_1 + b_1, \dots, a_n + b_n) \in I^n$ and $(ta_1, \dots, ta_n) \in I^n$ in properties (a06), (a07), (a08), (a09), (a10) and (a11) are necessary only when for interval I we use bounded interval I_1 .
- It is well known that additivity (a06) holds only for aggregation function *weighted arithmetic mean* $\text{WAM}_{[n]}^{\omega_{[n]}} : I^n \rightarrow I$, i.e., when aggregation function is a linear transformation on I^n . Namely, homogeneity (a09) of aggregation function follows from its monotonicity (a02) and additivity (a06).

3. Distortion functions

In this section we introduce a special aggregation function called distortion function. Definition of distortion functions, i.e., their monotonicity and boundary conditions are motivated by idea and methodology of construction of new aggregation functions presented in the next section. Namely, applying an arbitrary aggregation function on an arbitrary sequence of distortion functions, new aggregation function is constructed.

3.1. Definition of distortion function and MU properties

The concept of distortion functions as arguments of weighted arithmetic mean $WAM_{[n]}^{(O[n])}$ was introduced in [6].

Definition 6 (*Distortion function*). Distortion function is a monotonically nondecreasing function $f : I \rightarrow I$ which satisfies the boundary condition

$$f(0) = 0, \quad (3)$$

and, depending on the considered interval I , one of the following boundary conditions:

[I1] in the case of $I_1 = [0, 1]$,

$$f(1) = 1, \quad (4)$$

[I2] in the case of $I_2 = [0, \infty[$,

$$\lim_{x \rightarrow \infty} f(x) = \infty, \quad (5)$$

[I3] in the case of $I_3 = [0, \infty]$,

$$f(\infty) = \infty. \quad (6)$$

Dealing with subadditivity and superadditivity, as well as with continuity, monotonicity and boundary condition (6) of distortion functions and aggregation functions on interval I_3 , we need to precisely define these terms regarding value $\infty \in I_3$, as well as regarding Definition 3 of aggregation function and Definition 6 of distortion function, see [6]. For this purpose we introduce terms of MU-convex, MU-concave and MU-continuous distortion functions on interval I_3 .

Definition 7 (*MU-convexity on I_3*). Distortion function $f : [0, \infty] \rightarrow [0, \infty]$ is *monotonically-unlimited convex* on $[0, \infty]$, abbreviated *MU-convex*, if its restriction on interval $[0, \infty[$ is a convex function $f : [0, \infty[\rightarrow [0, \infty[$ (function with finite values) and $\lim_{x \rightarrow \infty} f(x) = \infty$. Additionally, function $f : [0, \infty] \rightarrow [0, \infty]$,

$$f(x) = \begin{cases} 0, & x < \infty \\ \infty, & x = \infty \end{cases}$$

will be considered as MU-convex.

Definition 8 (*MU-concavity on I_3*). Distortion function $f : [0, \infty] \rightarrow [0, \infty]$ is *monotonically-unlimited concave* on $[0, \infty]$, abbreviated *MU-concave*, if its restriction on interval $[0, \infty[$ is a concave function $f : [0, \infty[\rightarrow [0, \infty[$ (function with finite values) and $\lim_{x \rightarrow \infty} f(x) = \infty$. Additionally, function $f : [0, \infty] \rightarrow [0, \infty]$,

$$f(x) = \begin{cases} 0, & x = 0 \\ \infty, & x > 0 \end{cases}$$

will be considered as MU-concave.

Definition 9 (*MU-continuity on I_3*). Distortion function $f : [0, \infty] \rightarrow [0, \infty]$ is *monotonically-unlimited continuous* on $[0, \infty]$, abbreviated *MU-continuous*, if for $b = \sup \{x \in [0, \infty] \mid f(x) < \infty\}$ holds $b > 0$, $f : [0, b[\rightarrow [0, \infty[$ is a continuous function, and $\lim_{x \rightarrow b} f(x) = \infty$.

Remark 4. From monotonicity and MU-continuity, for distortion function $f : [0, \infty] \rightarrow [0, \infty]$ and $b = \sup \{x \in [0, \infty] \mid f(x) < \infty\}$, see Definition 9, obviously follows $f(b) = \infty$.

The following two statements about reflection of convexity/ MU-convexity and concavity/ MU-concavity on continuity/ MU-continuity of distortion functions were proved in [6].

Theorem 1. Let $f : I \rightarrow I$ be a distortion function. If f is a convex function on the interval I_1 or I_2 , i.e., f is a MU-convex function on the interval I_3 , then the distortion function f is a continuous function on the interval I_1 or I_2 , i.e. f is a MU-continuous function on the interval I_3 , except maybe at the point 1 in the case of interval I_1 , and except in ∞ for function $f : [0, \infty] \rightarrow [0, \infty]$,

$$f(x) = \begin{cases} 0 & , \quad x < \infty \\ \infty & , \quad x = \infty \end{cases} \text{ on the interval } I_3.$$

Theorem 2. Let $f : I \rightarrow I$ be a distortion function. If f is a concave function on the interval I_1 or I_2 , or f is a MU-concave function on the interval I_3 , then the distortion function f is a continuous function on the interval I_1 or I_2 , or f is a MU-continuous function on the interval I_3 , except maybe at the point 0, and except for function $f : [0, \infty] \rightarrow [0, \infty]$, $f(x) = \begin{cases} 0 & , x = 0 \\ \infty & , x > 0 \end{cases}$ on the interval I_3 .

The following examples illustrate that convex distortion function on I_1 can have discontinuity at 1, that MU-convex distortion function on I_3 can have discontinuity at ∞ , and that concave and MU-concave distortion function can have discontinuity at 0 on any of the considered intervals.

Example 2.

- (a) Distortion function $f : [0, 1] \rightarrow [0, 1]$, $f(x) = \begin{cases} \frac{1}{2}x^2 & , x < 1 \\ 1 & , x = 1 \end{cases}$ is convex, but has discontinuity at the point 1.
- (b) Distortion function $f : [0, \infty] \rightarrow [0, \infty]$, $f(x) = \begin{cases} 0 & , x < \infty \\ \infty & , x = \infty \end{cases}$ is MU-convex, but has discontinuity at the point ∞ .
Namely, it is only MU-convex distortion function with discontinuity at ∞ .
- (c) Distortion function $f : I \rightarrow I$, $f(x) = \begin{cases} 0 & , x = 0 \\ \frac{1}{2}x + \frac{1}{2} & , x > 0 \end{cases}$ is concave on I_1 and I_2 , i.e. MU-concave on I_3 , but has discontinuity at the point 0. ✓

3.2. Subadditivity and superadditivity

The next two theorems give the connection between the concavity of distortion function and its superadditivity, and between the convexity of distortion function and its subadditivity.

Theorem 3. Let $f : I \rightarrow I$ be a distortion function which is convex in the cases I_1 and I_2 , or MU-convex in the case I_3 . Function f is then superadditive, i.e., for all $x, y \in I$ it holds that

$$f(x + y) \geq f(x) + f(y).$$

Proof. In the case of interval I_3 and MU-convex distortion function f , for $x = \infty$ or $y = \infty$, because of boundary condition $f(\infty) = \infty$ equality $f(x + y) = f(\infty) = f(x) + f(y)$ comes down to obvious equalities $\infty = f(x) + \infty$ or $\infty = \infty + f(y)$. Because of that, we will further observe just $x, y \in I_1$ for which is $x + y \in I_1$, or $x, y \in I_2$. Let $I = I_1$ or $I = I_2$. Let us first prove that

$$\forall t \in [0, 1], \forall x \in I, f(tx) \leq tf(x). \quad [D1]$$

Because of convexity of function f on I and because of boundary condition $f(0) = 0$, for each $t \in [0, 1]$ and arbitrary $x \in I$ it holds that

$$f(tx) = f((1 - t) \cdot 0 + tx) \leq (1 - t) \cdot f(0) + tf(x) = tf(x).$$

For arbitrary $x, y \in I$ satisfying $x + y \in I_1$ in the case of interval I_1 , the proof of superadditivity follows. Using twice inequality [D1] with $t = \frac{x}{x+y}$ and $t = \frac{y}{x+y}$ we obtain

$$\begin{aligned} f(x + y) &= \frac{x+y}{x+y} f(x + y) = \frac{x}{x+y} f(x + y) + \frac{y}{x+y} f(x + y) \\ &\stackrel{[D1]}{\geq} f\left(\frac{x}{x+y}(x + y)\right) + f\left(\frac{y}{x+y}(x + y)\right) \\ &= f(x) + f(y). \end{aligned}$$

Inequality $f(x + y) \geq f(x) + f(y)$ is proved for all $x, y \in I$, and for all cases of considered intervals I . □

Theorem 4. Let $f : I \rightarrow I$ be a distortion function which is concave in the cases I_1 and I_2 , i.e., MU-concave in the case I_3 . Function f is then subadditive, i.e., for all $x, y \in I$ it holds that

$$f(x + y) \leq f(x) + f(y).$$

Proof. Proof is analogous to proof of Theorem 3. □

In the following examples we illustrate Theorems 3 and 4.

Example 3. Let $\alpha > 0$ and $k > 0$, and let function $f_{k,\alpha} : I \rightarrow I$ be defined by $f_{k,\alpha}(x) = kx^\alpha$, $x \in I$, where I is any of the considered intervals.

- (a) In the case I_1 , function $f_{1,\alpha}$ defined by
 $f_{1,\alpha}(x) = x^\alpha, x \in I$ is a distortion function.
 (b) In the cases I_2 and $I_3, f_{k,\alpha}$ is a distortion function.

Additionally, for any $\alpha \in [1, \infty[, f_{k,\alpha}$ is a convex/MU-convex function, and $f_{k,\alpha}$ is superadditive by Theorem 3. On the other hand, for any $\alpha \in]0, 1], f_{k,\alpha}$ is a concave/ MU-concave function, and $f_{k,\alpha}$ is subadditive by Theorem 4. ✓

Example 4. Let $\alpha > 0$ and $k > 0$, and let function $f_{k,\alpha} : I \rightarrow I$ be defined by $f_{k,\alpha}(x) = ke^{2x} - k, x \in I$.

- (a) In the case of the interval I_1 , function $f_{\frac{1}{e^2-1},\alpha}$ defined by

$$f_{\frac{1}{e^2-1},\alpha}(x) = \frac{1}{e^\alpha - 1} e^{2x} - \frac{1}{e^\alpha - 1}, x \in I$$

is a distortion function.

- (b) In the cases of intervals I_2 and $I_3, f_{k,\alpha}$ is a distortion function.

Beside this, for each $\alpha > 0$ and each $k > 0, f_{k,\alpha}$ is a convex/ MU-convex function, and it follows from Theorem 3 that $f_{k,\alpha}$ is superadditive. ✓

Example 5. Let $\alpha > 0$ and $k > 0$, and let function $f_{k,\alpha} : I \rightarrow I$ be defined by $f_{k,\alpha}(x) = k \cdot \ln(\alpha x + 1), x \in I$.

- (a) For interval I_1 , function $f_{\frac{1}{\ln(\alpha+1)},\alpha}$ defined by

$$f_{\frac{1}{\ln(\alpha+1)},\alpha}(x) = \frac{1}{\ln(\alpha+1)} \ln(\alpha x + 1), x \in I$$

is a distortion function.

- (b) In the cases of intervals I_2 and $I_3, f_{k,\alpha}$ is a distortion function.

Additionally, for each $\alpha > 0$ and each $k > 0, f_{k,\alpha}$ is a concave/ MU-concave function, and it follows from Theorem 4 that $f_{k,\alpha}$ is subadditive. ✓

Superadditivity or subadditivity of distortion functions follows not just from their convexity or concavity, but also from their boundary conditions, and from that distortion functions are monotonically nondecreasing. Convexity of monotone functions generally does not imply their superadditivity, and concavity of monotone functions generally does not imply their subadditivity. The following example illustrates that convex function can be also subadditive.

Example 6. Let $\alpha < 0$, and let function $g_\alpha :]0, \infty[\rightarrow]0, \infty[$ be defined by $g_\alpha(x) = x^\alpha, x \in]0, \infty[$. Since $g_\alpha''(x) = \alpha(\alpha - 1)x^{\alpha-2} > 0$ holds for each $x \in]0, \infty[$, it follows that g is a convex function. We will prove that g_α is a subadditive function. For $\alpha < 0$, i.e., $-\alpha > 0$ we obtain $g_\alpha(x+y) \leq g_\alpha(x) + g_\alpha(y) \iff (x+y)^\alpha \leq x^\alpha + y^\alpha \iff \frac{1}{(x+y)^{-\alpha}} \leq \frac{1}{x^{-\alpha}} + \frac{1}{y^{-\alpha}} = \frac{x^{-\alpha} + y^{-\alpha}}{x^{-\alpha}y^{-\alpha}} \iff x^{-\alpha}y^{-\alpha} \leq (x^{-\alpha} + y^{-\alpha})(x+y)^{-\alpha} = x^{-\alpha}(x+y)^{-\alpha} + (x+y)^{-\alpha}y^{-\alpha}$, and the last inequality is true for all $x, y \in]0, \infty[$ and each $-\alpha > 0$, since from $b^{-\alpha} \leq (a+b)^{-\alpha}, a^{-\alpha} \leq (a+b)^{-\alpha}$, follows that $a^{-\alpha}b^{-\alpha} \leq a^{-\alpha}(a+b)^{-\alpha}, a^{-\alpha}b^{-\alpha} \leq (a+b)^{-\alpha}b^{-\alpha}$. ✓

4. Aggregation of triangle of distortion functions

In this section we introduce an original method for a construction of new extended aggregation function by applying an extended aggregation function on the initial system (triangle) of distortion functions. Also, some properties of constructed extended aggregation function are discussed. Namely, the properties of constructed extended aggregation function depend on the properties of initial distortion functions, and the properties of applied initial extended aggregation function. The main interest is on subadditivity, superadditivity and other properties discussed in Sections 5 and 6 which are relevant for the following subjects:

- (1) construction of new distance functions by applying aggregation function on the sequence of initial distance functions, see for example [1,2,4,7–9],
- (2) construction of new fuzzy measure by applying aggregation function on the sequence of initial fuzzy measures, see for example [6].

4.1. Aggregation of distance functions and fuzzy measures

Construction or choice of appropriated distance functions and fuzzy measures plays significant role in many decision making tasks. For example, in [2,7–9] distance functions were constructed by applying aggregation function, and they are used in image segmentation. Similarly, applying aggregation function new fuzzy measure can be constructed, see [6], and applied in decision making processes as well as for other applications. In the next two definitions and theorems follows the brief description of mentioned constructions, for some details and proofs see [6,8].

Definition 10 (Distance function). Let $X \neq \emptyset$. Function $d : X^2 \rightarrow [0, \infty[$ is a *distance function* if $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$.

Definition 11 (Fuzzy measure). Let $I = I_1, I = I_2$ or $I = I_3$. For $X \neq \emptyset$ and σ -algebra $\mathcal{A}$, function $m : \mathcal{A} \rightarrow I$ is a *fuzzy measure* on $\mathcal{A}$ if $m(\emptyset) = 0$ and $m(A) \leq m(B)$ for all $A \subseteq B, A, B \in \mathcal{A}$.

Theorem 5. For a fixed $n \in \mathbb{N}$, let $d_i : X^2 \rightarrow [0, \infty[, i \in \{1, \dots, n\}$, be a sequence of distance functions on $X \neq \emptyset$, and let $A_{[n]} : [0, \infty[^n \rightarrow [0, \infty[$ be an arbitrary n -ary aggregation function. Function $d_{[n]} : X^2 \rightarrow [0, \infty[$ defined by

$$d_{[n]}(x, y) = A_{[n]}(d_1(x, y), \dots, d_n(x, y)), \quad x, y \in X$$

is a distance function on X .

Theorem 6. For a fixed $n \in \mathbb{N}$, let $m_i : \mathcal{A} \rightarrow I, i \in \{1, \dots, n\}$, be a sequence of fuzzy measures on σ -algebra $\mathcal{A}$, and let $A_{[n]} : I^n \rightarrow I$ be an arbitrary n -ary aggregation function. Function $m_{[n]} : \mathcal{A} \rightarrow I$ defined by

$$m_{[n]}(F) = A_{[n]}(m_1(F), \dots, m_n(F)), \quad F \in \mathcal{A}$$

is a fuzzy measure on $\mathcal{A}$.

Additional properties of constructed distance functions or fuzzy measures depend on properties of applied aggregation function and properties of distance functions or fuzzy measures used in construction.

4.2. Aggregation of triangle of distortion functions

Let again I denote one of the intervals $I_1 = [0, 1], I_2 = [0, \infty[$ or $I_3 = [0, \infty]$ from Section 2. In the following two definitions we introduce the notions of triangle of distortion functions and aggregated triangle of distortion functions.

Definition 12. For each $n \in \mathbb{N}$, let $\mathcal{F}_{[n]} = (f_{n,1}, \dots, f_{n,n})$ be an n -tuple of distortion functions $f_{n,i} : I \rightarrow I, i \in \{1, \dots, n\}$. System $\mathcal{F} = \{\mathcal{F}_{[n]} | n \in \mathbb{N}\}$ of such functions is called *triangle of distortion functions*.

Definition 13. Let $\mathcal{F} = \{\mathcal{F}_{[n]} | n \in \mathbb{N}\}$ be an triangle of distortion functions, and let $A : \bigcup_{n=1}^{\infty} I^n \rightarrow I$ be an extended aggregation function with n -ary restrictions - aggregation functions $A_{[n]} : I^n \rightarrow I, n \in \mathbb{N}$. Function $A_{[n]}^{\mathcal{F}_{[n]}} : I^n \rightarrow I$ defined by

$$A_{[n]}^{\mathcal{F}_{[n]}}(a_1, \dots, a_n) = A_{[n]}(f_{n,1}(a_1), \dots, f_{n,n}(a_n)), \quad (7)$$

$(a_1, \dots, a_n) \in I^n$, is called $A_{[n]}$ -aggregated distortion functions $\mathcal{F}_{[n]}$. The corresponding function $A^{\mathcal{F}} : \bigcup_{n=1}^{\infty} I^n \rightarrow I$ with restrictions $A_{[n]}^{\mathcal{F}_{[n]}}$ is called A -aggregated triangle of distortion functions $\mathcal{F}$.

Proposition 1. For each $n \in \mathbb{N}$, function $A_{[n]}^{\mathcal{F}_{[n]}}$ from Definition 13 is an n -ary aggregation function, and function $A^{\mathcal{F}}$ is an extended aggregation function.

Proof. Boundary conditions (B0) and (B1) from Definition 1 are satisfied for $A_{[n]}^{\mathcal{F}_{[n]}}$ because they are satisfied for aggregation function $A_{[n]}$, and distortion functions $f_{n,i}, i \in \{1, \dots, n\}$, satisfy boundary condition (3) and one of the boundary conditions (4), (5) or (6), depending on the considered interval I . On the other hand, for a function $A_{[n]}^{\mathcal{F}_{[n]}}$, monotonicity (a02) from Definition 1 follows because $A_{[n]}^{\mathcal{F}_{[n]}}$ is a composition of monotone nondecreasing aggregation function $A_{[n]}$ and monotone nondecreasing distortion functions $f_{n,i}, i \in \{1, \dots, n\}$. $\square$

In the following example, we present an aggregation function, and the corresponding extended aggregation function, which is a special case of $A_{[n]}^{\mathcal{F}_{[n]}}$ defined by (7). This aggregation function, called *weighted arithmetic mean of distorted arguments* and denoted as $\text{WAMDA}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}}$ was introduced in [6], where its properties are also discussed. One more special type of $\text{WAMDA}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}}$ aggregation function, called *weighted arithmetic mean of powers* and denoted as $\text{WAMP}_{[n]}^{\omega_{[n]}, \lambda_{[n]}}$, was introduced and discussed in [1].

Example 7. Let $\omega = \{\omega_{[n]} \mid n \in \mathbb{N}\}$ be a triangle of weights, and let $\mathcal{F} = \{\mathcal{F}_{[n]} \mid n \in \mathbb{N}\}$ be a triangle of distortion functions. For each $n \in \mathbb{N}$, function $\text{WAMDF}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}} : I^n \rightarrow I$ defined by

$$\text{WAMDF}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}}(a_1, \dots, a_n) = \sum_{i=1}^n \omega_{n,i} f_{n,i}(a_i), \quad (a_1, \dots, a_n) \in I^n \quad (8)$$

is an n -ary aggregation function which we call *weighted arithmetic mean of distortion function*. The corresponding extended aggregation function $\text{WAMDF}^{\omega, \mathcal{F}} : \bigcup_{n=1}^{\infty} I^n \rightarrow I$ is defined by restrictions $\text{WAMDF}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}}$, $n \in \mathbb{N}$. In general case, aggregation function $\text{WAMDF}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}}$ does not have any of the properties from Definitions 3 and 4. With some additional assumption on parameters $\omega_{[n]}$ and $\mathcal{F}_{[n]}$, some of the mentioned properties can be satisfied, see [6].

A special case of $\text{WAMDF}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}}$ aggregation function is when distortion functions $f_{n,i}$, $i \in \{1, \dots, n\}$, are power functions with positive powers. Let $\omega_{[n]}$ be an n -tuple of nonnegative real numbers from some triangle of weights $\omega = \{\omega_{[n]} \mid n \in \mathbb{N}\}$, and let $\lambda_{[n]} = (\lambda_{n,1}, \dots, \lambda_{n,n}) \in]0, \infty[^n$ be an n -tuple of positive real numbers. We call the aggregation function $\text{WAMP}_{[n]}^{\omega_{[n]}, \lambda_{[n]}} : I^n \rightarrow I$ defined by

$$\text{WAMP}_{[n]}^{\omega_{[n]}, \lambda_{[n]}}(a_1, \dots, a_n) = \sum_{i=1}^n \omega_{n,i} a_i^{\lambda_{n,i}}, \quad (a_1, \dots, a_n) \in I^n \quad (9)$$

weighted arithmetic mean of powers, and the corresponding extended aggregation function $\text{WAMP}^{\omega, \lambda} : \bigcup_{n=1}^{\infty} I^n \rightarrow I$ is defined by restrictions $\text{WAMP}_{[n]}^{\omega_{[n]}, \lambda_{[n]}}$, $n \in \mathbb{N}$ and the triangle of powers $\lambda = \{\lambda_{[n]} \mid n \in \mathbb{N}\}$. $\text{WAMP}_{[n]}^{\omega_{[n]}, \lambda_{[n]}}$ aggregation function is introduced in [1], where the properties of this function are also discussed. For example, $\text{WAMP}_{[n]}^{\omega_{[n]}, \lambda_{[n]}}$ is a continuous aggregation function, subadditive in the case when $\lambda_{n,i} < 1$ for all $i \in \{1, \dots, n\}$, and superadditive in the case when $\lambda_{n,i} > 1$ for all $i \in \{1, \dots, n\}$. ✓

Remark 5. In general case, $A_{[n]}$ -aggregated distortion functions $\mathcal{F}_{[n]}$, for arbitrary parameters $\mathcal{F}_{[n]}$ have none of properties from Definitions 3 and 4. Namely, as its special case, $\text{WAMDF}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}}$ aggregation function also has none of these properties.

4.3. Properties of aggregated triangle of distortion functions

Further, we discuss under which additional assumption on distortion functions $\mathcal{F}_{[n]}$ certain properties are satisfied. Let I be any of the considered intervals I_1, I_2 or I_3 . For $n \geq 2$, let $f_{n,i} : I \rightarrow I$, $i \in \{1, \dots, n\}$, be an arbitrary sequence of distortion functions and $\mathcal{F}_{[n]} = (f_{n,1}, \dots, f_{n,n})$, and let $A_{[n]}$ be an arbitrary n -ary aggregation function. Let $A_{[n]}^{\mathcal{F}_{[n]}}$ be an $A_{[n]}$ -aggregated distortion functions $\mathcal{F}_{[n]}$ defined by (7).

(a03) If the aggregation function $A_{[n]}$ and all the distortion functions $f_{n,i} : I \rightarrow I$, $i \in \{1, \dots, n\}$, are continuous, then, obviously, the aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ is continuous as a composition of continuous functions. Beside that, for convex and concave distortion functions we can obtain some additional conclusions about the continuity of $A_{[n]}^{\mathcal{F}_{[n]}}$. This conclusions are formulated in the following theorem.

Theorem 7. For any n -ary aggregation function $A_{[n]}$ and arbitrary sequence of distortion functions $f_{n,i} : I \rightarrow I$, $i \in \{1, \dots, n\}$, the following statements hold for aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ defined by (7).

- (a) If $A_{[n]}$ is a continuous aggregation function in all points $(a_1, \dots, a_n) \in I^n$ for which $a_i \neq 0$ for all $i \in \{1, \dots, n\}$, and if all distortion functions $f_{n,i}$, $i \in \{1, \dots, n\}$, are concave in the case of intervals I_1 and I_2 , or MU-concave in the case of interval I_3 , then aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ is also continuous in all points $(a_1, \dots, a_n) \in I^n$ for which $a_i \neq 0$ for all $i \in \{1, \dots, n\}$.
- (b) If $A_{[n]}$ is a continuous aggregation function in all points $(a_1, \dots, a_n) \in I^n$ for which
 - $a_i \neq 1$ for all $i \in \{1, \dots, n\}$ in the case of interval I_1 ,

• $a_i \neq \infty$ for all $i \in \{1, \dots, n\}$ in the case of interval I_3 ,
 and if all distortion functions $f_{n,i}, i \in \{1, \dots, n\}$ are convex in the case of intervals I_1 and I_2 , or MU-convex in the case of interval I_3 ,
 then aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ is also continuous in all referred points.

Proof.

- (a) By [Theorem 2](#), each of concave or MU-concave distortion functions $f_{n,i}, i \in \{1, \dots, n\}$ is continuous or MU-continuous except maybe in 0. Let $S =]0, 1]^n$ in the case of interval I_1 , $S =]0, \infty[^n$ in the case of interval I_2 , or $S =]0, \infty]^n$ in the case of interval I_3 . Then, for the aggregation function $A_{[n]}$ which is continuous on the set S , the aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ is also continuous on S as a composition of continuous functions $A_{[n]}$ and $f_{n,i}, i \in \{1, \dots, n\}$.
- (b) By [Theorem 1](#), each of convex or MU-convex distortion functions $f_{n,i}, i \in \{1, \dots, n\}$, is continuous or MU-continuous except maybe in 1 in the case of interval I_1 , or except maybe in ∞ in the case of interval I_3 . Let $S = [0, 1]^n$ in the case of interval I_1 , or $S = [0, \infty]^n$ in the case of intervals I_2 and I_3 . Then, for the aggregation function $A_{[n]}$ which is continuous on the set S , the aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ is also continuous on S as a composition of continuous functions $A_{[n]}$ and $f_{n,i}, i \in \{1, \dots, n\}$. $\square$

We continue with the examination of properties of aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$. Regarding the properties (a04) and (a05), generally we can state the following.

Remark 6.

- (a04) It is obvious that $A_{[n]}^{\mathcal{F}_{[n]}}$ is symmetric only if $A_{[n]}$ is a symmetric aggregation function and $f_{n,i} = f$ for some distortion function f and all $i \in \{1, \dots, n\}$.
- (a05) Aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ is idempotent only under very restrictive conditions on aggregation function $A_{[n]}$ and distortion functions $\mathcal{F}_{[n]}$.

5. Subadditivity and superadditivity of aggregated distortion functions

In this section we investigate subadditivity and superadditivity of the aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ defined in [Section 4](#), i.e., we analyze the conditions under which $A_{[n]}^{\mathcal{F}_{[n]}}$ is subadditive or superadditive. Generally we can state the following.

Remark 7.

- (a06) Aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ is additive only if it reduces on weighted arithmetic mean function $\text{WAM}_{[n]}^{(\omega_{[n]})}$, see [Remark 3](#). (iii) and [Example 1](#).
- (a07) Generally, aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ is not subadditive. In considerations bellow, we discuss some additional conditions on aggregation function $A_{[n]}$ and distortion functions $f_{n,i} : I \rightarrow I, i \in \{1, \dots, n\}$, under which aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ is subadditive.
- (a08) For the question of superadditivity of function $A_{[n]}^{\mathcal{F}_{[n]}}$, the same statement holds as for subadditivity (a07).

The following theorem gives one sufficient condition for subadditivity and superadditivity of $A_{[n]}^{\mathcal{F}_{[n]}}$ aggregation function. For $\text{WAMP}_{[n]}^{(\omega_{[n]}, \lambda_{[n]})}$, the aggregation function is in [Example 7](#), while the conditions for subadditivity and superadditivity of $\text{WAMP}_{[n]}^{(\omega_{[n]}, \lambda_{[n]})}$ function, as a special case of $A_{[n]}^{\mathcal{F}_{[n]}}$ function are discussed in [\[1\]](#).

Theorem 8. Let $A_{[n]}$ be an arbitrary n -ary aggregation function, and let $f_{n,i} : I \rightarrow I, i \in \{1, \dots, n\}$, be an arbitrary sequence of distortion functions. Then for aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ defined by [\(7\)](#) the following holds.

- (a) If $A_{[n]}$ is a subadditive aggregation function, i.e., has property (a07) from [Definition 4](#), and if all distortion functions $f_{n,i}, i \in \{1, \dots, n\}$, are subadditive, i.e., $f_{n,i}(a + b) \leq f_{n,i}(a) + f_{n,i}(b)$ for all $a, b, a + b \in I$ and each $i \in \{1, \dots, n\}$, then $A_{[n]}^{\mathcal{F}_{[n]}}$ is also a subadditive aggregation function (satisfies property (a07) from [Definition 4](#)).

- (b) If $A_{[n]}$ is a superadditive aggregation function, i.e., has property (a08) from Definition 4, and if all distortion functions $f_{n,i}$, $i \in \{1, \dots, n\}$, are superadditive, i.e., $f_{n,i}(a+b) \geq f_{n,i}(a) + f_{n,i}(b)$ for all $a, b, a+b \in I$ and each $i \in \{1, \dots, n\}$, then $A_{[n]}^{\mathcal{F}[n]}$ is also a superadditive aggregation function (satisfies property (a08) from Definition 4).

Proof.

- (a) For arbitrary n -tuples $(a_1, \dots, a_n) \in I^n$, $(b_1, \dots, b_n) \in I^n$, which satisfy $(a_1 + b_1, \dots, a_n + b_n) \in I^n$, applying monotonicity (a02) and subadditivity (a07) of aggregation function $A_{[n]}$, subadditivity

$$f_{n,i}(a_i + b_i) \leq f_{n,i}(a_i) + f_{n,i}(b_i) \quad [*]$$

of distortion functions $f_{n,i}$, it follows

$$\begin{aligned} A_{[n]}^{\mathcal{F}[n]}(a_1 + b_1, \dots, a_n + b_n) &= A_{[n]}(f_{n,1}(a_1 + b_1), \dots, f_{n,n}(a_n + b_n)) \\ &\stackrel{[*], (a02)}{\leq} A_{[n]}(f_{n,1}(a_1) + f_{n,1}(b_1), \dots, f_{n,n}(a_n) + f_{n,n}(b_n)) \\ &\stackrel{(a07)}{\leq} A_{[n]}(f_{n,1}(a_1), \dots, f_{n,n}(a_n)) + A_{[n]}(f_{n,1}(b_1), \dots, f_{n,n}(b_n)) \\ &= A_{[n]}^{\mathcal{F}[n]}(a_1, \dots, a_n) + A_{[n]}^{\mathcal{F}[n]}(b_1, \dots, b_n). \end{aligned}$$

- (b) The proof is analogous to the proof of statement (a), with $\geq$ inequality instead of $\leq$, and with superadditivity (a08) instead of subadditivity (a07) of aggregation function $A_{[n]}$. $\square$

In the following theorem, one more sufficient condition for subadditivity and superadditivity of $A_{[n]}^{\mathcal{F}[n]}$ aggregation function is proved. The following sufficient conditions also have reflection on the question of continuity of the aggregation function $A_{[n]}^{\mathcal{F}[n]}$.

Theorem 9. For arbitrary n -ary aggregation function $A_{[n]}$ and arbitrary sequence of distortion functions $f_{n,i} : I \rightarrow I$, $i \in \{1, \dots, n\}$, the following statements hold for aggregation function $A_{[n]}^{\mathcal{F}[n]}$ defined by (7).

- (a) If $A_{[n]}$ is a subadditive aggregation function, i.e., has property (a07) from Definition 4, and if all distortion functions $f_{n,i}$, $i \in \{1, \dots, n\}$, are concave in the case of intervals I_1 and I_2 , or MU-concave in the case of interval I_3 , then aggregation function $A_{[n]}^{\mathcal{F}[n]}$ is subadditive (satisfies property (a07) from Definition 4).
- (b) If $A_{[n]}$ is a superadditive aggregation function, i.e., has property (a08) from Definition 4, and if all distortion functions $f_{n,i}$, $i \in \{1, \dots, n\}$, are convex in the case of intervals I_1 and I_2 , or MU-convex in the case of interval I_3 , then aggregation function $A_{[n]}^{\mathcal{F}[n]}$ is superadditive (satisfies property (a08) from Definition 4).

Proof.

- (a) By Theorem 4, concave or MU-concave distortion functions $f_{n,i}$, $i \in \{1, \dots, n\}$, are subadditive on considered interval I , and therefore, by Theorem 8, aggregation function $A_{[n]}^{\mathcal{F}[n]}$ is subadditive.
- (b) By Theorem 3, convex or MU-convex distortion functions $f_{n,i}$, $i \in \{1, \dots, n\}$, are subadditive on considered interval I , and therefore, by Theorem 8, aggregation function $A_{[n]}^{\mathcal{F}[n]}$ is superadditive. $\square$

6. Some additional properties of aggregated distortion functions

Further we continue the examination of the rest of properties of n -ary aggregation function $A_{[n]}^{\mathcal{F}[n]}$ listed in Definition 4.

- (a09) For aggregation function $\text{WAMP}_{[n]}^{\omega_{[n]}, \lambda_{[n]}}$ from Example 7 defined by (9), it is proved in [1] that $\text{WAMP}_{[n]}^{\omega_{[n]}, \lambda_{[n]}}$ aggregation function is neither positive subhomogenous nor positive superhomogenous, and that there are no interesting classes of parameters $\omega_{[n]}$ and $\lambda_{[n]}$ (except some trivial cases, for example for weighted arithmetic mean $\text{WAM}_{[n]}^{\omega_{[n]}}$) for which $\text{WAMP}_{[n]}^{\omega_{[n]}, \lambda_{[n]}}$ aggregation function has the mentioned properties. Because the aggregation function $\text{WAMP}_{[n]}^{\omega_{[n]}, \lambda_{[n]}}$ is a special case of $A_{[n]}^{\mathcal{F}[n]}$ aggregation function, the same conclusions about properties (a09), (a10) and (a11) hold for $A_{[n]}^{\mathcal{F}[n]}$.

(a10) See (a09).

(a11) See (a09).

(a12) It is obvious under which additional assumptions $A_{[n]}^{\mathcal{F}_{[n]}}$ aggregation function has the property (a12). Namely, if aggregation function $A_{[n]}$ has the property (a12) and each of distortion functions $f_{n,i}$, $i \in \{1, \dots, n\}$, satisfy $f_{n,i}(a_i) = 0$ only for $a_i = 0$, then aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ has the property (a12) because

$$\begin{aligned} A_{[n]}^{\mathcal{F}_{[n]}}(a_1, \dots, a_n) &= A_{[n]}(f_{n,1}(a_1), \dots, f_{n,n}(a_n)) = 0 \\ &\Rightarrow \forall i \in \{1, \dots, n\}, f_{n,i}(a_i) = 0 \\ &\Rightarrow \forall i \in \{1, \dots, n\}, a_i = 0. \end{aligned}$$

(a13) Generally, aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ does not have the property (a13) (see [Example 8](#)). But, analogously as for the property (a12), if an aggregation function $A_{[n]}$ has the property (a13) and some of distortion functions $f_{n,i}$, $i \in \{1, \dots, n\}$, satisfies $f_{n,i}(a_i) = 0$ only for $a_i = 0$, then the aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ has the property (a13) because

$$\begin{aligned} A_{[n]}^{\mathcal{F}_{[n]}}(a_1, \dots, a_n) &= A_{[n]}(f_{n,1}(a_1), \dots, f_{n,n}(a_n)) = 0 \\ &\Rightarrow \exists i_0 \in \{1, \dots, n\}, f_{n,i_0}(a_{i_0}) = 0, \end{aligned}$$

and then $a_{i_0} = 0$ holds.

The following example illustrates that, in general case, aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ does not have properties (a12) and (a13).

Example 8. Let $n \geq 2$, and let I be any of considered intervals I_1 , I_2 or I_3 . Further, let $\omega_{[n]} = (\omega_{n,1}, \dots, \omega_{n,n})$ with $\omega_{n,i} = \frac{1}{n}$, $i \in \{1, \dots, n\}$, and let functions $f_{n,i} : I \rightarrow I$ be defined by

$$f_{n,i}(a) = \begin{cases} 0, & a \leq 0.5 \\ 2a - 1, & a > 0.5 \end{cases}$$

for each $i \in \{1, \dots, n\}$. Obviously, $f_{n,i}$ are distortion functions. With coefficients $\omega_{[n]}$, aggregation function $A_{[n]} = \text{WAM}_{[n]}^{\omega_{[n]}}$ from [Example 1](#) and distortion functions $\mathcal{F}_{[n]} = (f_{n,1}, \dots, f_{n,n})$, applying (7) we obtain $\text{WAMDF}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}} : I^n \rightarrow I$ aggregation function from [Example 7](#), defined by

$$\text{WAMDF}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}}(a_1, \dots, a_n) = \frac{1}{n} \sum_{i=1}^n f_{n,i}(a_i), \quad (a_1, \dots, a_n) \in I^n, \quad (10)$$

which is also a $\text{WAM}_{[n]}^{\omega_{[n]}}$ -aggregated distortion functions $\mathcal{F}_{[n]}$ from [Definition 13](#). For function $\text{WAMDF}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}}$ properties (a12) and (a13) are obviously not satisfied because, for example, for $a_i = 0.1 \neq 0$, $i \in \{1, \dots, n\}$, holds $\text{WAMDF}_{[n]}^{\omega_{[n]}, \mathcal{F}_{[n]}}(a_1, \dots, a_n) = 0$. $\blacktriangleright$

Let us consider the last property (a14) from [Definition 4](#) for the aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$.

(a14) As we will see in the following discussion, the aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ defined by (7) does not have property (a14) in the general case. But, depending on the considered interval I , the applied aggregation function $A_{[n]}$ and the properties of distortion functions $\mathcal{F}_{[n]}$, property (a14) can be satisfied.

[I1] For the interval I_1 , the aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ does not have the property (a14). For example, for $n = 2$, using

$A_{[2]} = \text{WAM}_{[2]}^{\omega_{[2]}}$ aggregation function from [Example 1](#) with coefficients $\omega_{[2]} = (0.5, 0.5)$, and using distortion functions

$$f_{2,1}(x) = x, \quad x \in [0, 1], \quad f_{2,2}(a) = \begin{cases} 2a, & a \leq 0.5 \\ 1, & a > 0.5 \end{cases},$$

we obtain

$$A_{[2]}^{\mathcal{F}_{[2]}}(0.5, 1) = 0.5 \cdot f_{2,1}(0.5) + 0.5 \cdot f_{2,2}(1) = 0.5 \cdot 0.5 + 0.5 \cdot 1 = 0.75 < 1,$$

for $a_2 = 1$.

For certain types of distortion functions $\mathcal{F}_{[n]}$ and applied aggregation functions $A_{[n]}$, property (a14) can be satisfied. One sufficient condition for property (a14) on interval I_1 is formulated in [Proposition 2](#) below.

[I2/I3] For intervals I_2 and I_3 , aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ does not have property (a14). For example, for $n = 2$, using $A_{[2]} = \text{WAM}_{[2]}^{(\omega_{[2]})}$ aggregation function from [Example 1](#) with coefficients $\omega_{[2]} = (0, 1)$, and using distortion functions $f_{2,1}(x) = f_{2,2}(x) = x, x \in I$, we obtain

$$A_{[2]}^{\mathcal{F}_{[2]}}(a_1, a_2) = 0 \cdot f_{2,1}(a_1) + 1 \cdot f_{2,2}(a_2) = 0 \cdot a_1 + 1 \cdot a_2 = a_2,$$

so that

[I2] for I_2 and $(b_1, b_2) =]\infty, 1[$ we have

$$\lim_{(a_1, a_2) \rightarrow]\infty, 1[} A_{[2]}^{\mathcal{F}_{[2]}}(a_1, a_2) = \lim_{(a_1, a_2) \rightarrow]\infty, 1[} a_2 = 1 < \infty$$

for $b_1 = \infty$,

[I3] for I_3 and $(a_1, a_2) =]\infty, 1[$ we have

$$A_{[2]}^{\mathcal{F}_{[2]}}(\infty, 1) = 1 < \infty$$

for $a_1 = \infty$.

For certain types of distortion functions $\mathcal{F}_{[n]}$ and applied aggregation functions $A_{[n]}$, property (a14) can be satisfied. One sufficient condition for property (a14) on intervals I_2 and I_3 is formulated in [Proposition 3](#) below.

Proposition 2. Let aggregation function $A_{[n]} : [0, 1]^n \rightarrow [0, 1]$ satisfy property (a14) from [Definition 4](#), and let for each of distortion functions $f_{n,i} : [0, 1] \rightarrow [0, 1], i \in \{1, \dots, n\}$, hold that $f_{n,i}(x) = 1$ if and only if $x = 1$. Then the aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ defined by (7) satisfies property (a14).

Proof. From

$$A_{[n]}^{\mathcal{F}_{[n]}}(a_1, \dots, a_n) = A_{[n]}(f_{n,1}(a_1), \dots, f_{n,n}(a_n)) < 1$$

and property (a14) of applied aggregation function $A_{[n]}$ follows that inequality $f_{n,i}(a_i) < 1$ holds for each $i \in \{1, \dots, n\}$, and then $\forall i \in \{1, \dots, n\}, a_i < 1$, follows from the condition $f_{n,i}(x) = 1 \iff x = 1$ of distortion functions $f_{n,i}, i \in \{1, \dots, n\}$. $\square$

Proposition 3. Let I be one of the intervals I_2 or I_3 . Let aggregation function $A_{[n]} : I^n \rightarrow I$ satisfy property (a14) from [Definition 4](#). Let for each of distortion functions $f_{n,i} : I \rightarrow I, i \in \{1, \dots, n\}$, hold the following:

[I2] $\lim_{a_i \rightarrow b_i} f_{n,i}(a_i) = \infty$ if and only if $b_i = \infty$ in the case of interval $I_2 = [0, \infty[$,

[I3] $f_{n,i}(a_i) = \infty$ if and only if $a_i = \infty$ in the case of interval $I_3 = [0, \infty]$.

Then the aggregation function $A_{[n]}^{\mathcal{F}_{[n]}}$ defined by (7) also satisfies the property (a14).

Proof. Let aggregation function $A_{[n]} : I^n \rightarrow I$ satisfy property (a14).

[I2] In the case of interval I_2 , let for each $i \in \{1, \dots, n\}$, hold that $\lim_{a_i \rightarrow b_i} f_{n,i}(a_i) = \infty$ if and only if $b_i = \infty$. Then from

$$\lim_{(a_1, \dots, a_n) \rightarrow (b_1, \dots, b_n)} A_{[n]}^{\mathcal{F}_{[n]}}(a_1, \dots, a_n) = \lim_{(a_1, \dots, a_n) \rightarrow (b_1, \dots, b_n)} A_{[n]}(f_{n,1}(a_1), \dots, f_{n,n}(a_n)) < \infty$$

and property (a14) of applied aggregation function $A_{[n]}$ follows that $b_i < \infty$ for each of $i \in \{1, \dots, n\}$. From $b_i < \infty, i \in \{1, \dots, n\}$, and condition

$$\forall i \in \{1, \dots, n\}, \lim_{a_i \rightarrow b_i} f_{n,i}(a_i) = \infty \iff b_i = \infty,$$

follows that $b_i < \infty$ holds for each $i \in \{1, \dots, n\}$.

[I3] In the case of the interval I_3 , let for each $i \in \{1, \dots, n\}$ hold that $f_{n,i}(a_i) = \infty$ if and only if $a_i = \infty$. Then from

$$A_{[n]}^{\mathcal{F}_{[n]}}(a_1, \dots, a_n) = A_{[n]}(f_{n,1}(a_1), \dots, f_{n,n}(a_n)) < \infty$$

and property (a14) of applied aggregation function $A_{[n]}$ follows that for each $i \in \{1, \dots, n\}$ holds $f_{n,i}(a_i) < \infty$, and then from condition

$$\forall i \in \{1, \dots, n\}, \quad f_{n,i}(a_i) = \infty \iff a_i = \infty,$$

follows that $a_i < \infty$ for each $i \in \{1, \dots, n\}$. $\square$

7. Examples of the construction from Section 4

In this section we present some examples of the construction of new aggregation functions by applying some initial aggregation functions on the sequence of distortion functions, i.e., using the method from Section 4 summarized in Formula (7). Further, let I be again one of the intervals I_1 , I_2 or I_3 .

For the purpose of discussion in the examples listed in this section, we will consider some properties of well known aggregation functions minimum and maximum, see [29,31,36]. For each $n \geq 2$, n -ary aggregation functions $\min_{[n]}(a_1, \dots, a_n)$, $(a_1, \dots, a_n) \in I^n$, and $\max_{[n]}(a_1, \dots, a_n)$, $(a_1, \dots, a_n) \in I^n$, are basic fuzzy intersection and fuzzy union operators, see [46], and also basic and simple aggregation functions. For aggregation functions $\min_{[n]}$ and $\max_{[n]}$, let us consider some properties listed in Definitions 3 and 4.

- (a03) It is known and easy to prove that $\min_{[n]}$ and $\max_{[n]}$ aggregation functions are continuous on all of considered intervals I .
- (a07) In [8] it is proved that $\max_{[n]}$ aggregation function is subadditive on the interval I_1 , and the same proof is valid on the intervals I_2 and I_3 . That also implies that $\max_{[n]}$ cannot be superadditive, because subadditivity and superadditivity together mean additivity, but $\max_{[n]}$ cannot be an additive function - see Remark 3. (iii).
- (a08) The proof of superadditivity of $\min_{[n]}$ function is analogous to proof of subadditivity of $\max_{[n]}$ function. Because of superadditivity, $\min_{[n]}$ cannot be subadditive at the same time. As noted above, subadditivity and superadditivity together mean additivity, but $\min_{[n]}$ cannot be additive function - see Remark 3. (iii).

In the following examples, we will construct a new aggregation function applying in Formula (7) the aggregation functions $\min_{[n]}$ and $\max_{[n]}$ on some sequences of distortion functions, where choosing distortion functions with certain properties we obtain new aggregation functions with adequate properties. Let further $n \geq 2$.

Example 9. For arbitrary sequences $\alpha_i > 0, i \in \{1, \dots, n\}$ and $k_i > 0, i \in \{1, \dots, n\}$, of positive real numbers and n -tuples $\alpha_{[n]} = (\alpha_1, \dots, \alpha_n)$ and $K_{[n]} = (k_1, \dots, k_n)$, let functions $f_{k_i, \alpha_i} : I \rightarrow I, i \in \{1, \dots, n\}$, be defined by $f_{k_i, \alpha_i}(x) = k_i e^{\alpha_i x} - k_i, x \in I$, where I is any of the considered intervals, see Example 4. Denote $\mathcal{F}_{[n]} = (f_{k_1, \alpha_1}, \dots, f_{k_n, \alpha_n})$. As we concluded in Example 4,

- (a) in the case of interval I_1 , functions $f_{\frac{1}{e^{\alpha_i}-1}, \alpha_i}, i \in \{1, \dots, n\}$, are convex and superadditive distortion functions,
- (b) in the case of intervals I_2 and I_3 , for any $\alpha_i > 0$ and any $k_i > 0$, functions f_{k_i, α_i} are convex/ MU-convex and superadditive distortion functions.

Applying superadditive aggregation function $\min_{[n]}$ with distortion functions $\mathcal{F}_{[n]}$ in formula (7), we obtain the aggregation function

$$\min_{[n]}^{\mathcal{F}_{[n]}}(a_1, \dots, a_n) = \min_{[n]} \left(f_{\frac{1}{e^{\alpha_1}-1}, \alpha_1}(a_1), \dots, f_{\frac{1}{e^{\alpha_n}-1}, \alpha_n}(a_n) \right) \quad (11)$$

for $(a_1, \dots, a_n) \in [0, 1]^n$, in the case of interval I_1 , or aggregation function

$$\min_{[n]}^{\mathcal{F}_{[n]}}(a_1, \dots, a_n) = \min_{[n]} \left(f_{k_1, \alpha_1}(a_1), \dots, f_{k_n, \alpha_n}(a_n) \right) \quad (12)$$

for $(a_1, \dots, a_n) \in I^n$ and intervals I_2 and I_3 .

In all cases of intervals and parameters k_i and α_i , functions f_{k_i, α_i} are continuous/MU-continuous and convex/MU-convex. On the other hand, aggregation function $\min_{[n]}$ is continuous and superadditive, so that by Theorems 7 and 9. (b), aggregation function $\min_{[n]}^{\mathcal{F}_{[n]}}$ is continuous and superadditive. By above considerations, it is easy to see that $\min_{[n]}^{\mathcal{F}_{[n]}}$ does not have property (a12), but has property (a13). Also it does not have property (a14) because $\min_{[n]}$ does not have property (a14). $\blacktriangleright$

Example 10. For arbitrary sequences $\alpha_i > 0, i \in \{1, \dots, n\}$, and $k_i > 0, i \in \{1, \dots, n\}$ of positive real numbers and n -tuples $\alpha_{[n]} = (\alpha_1, \dots, \alpha_n)$ and $K_{[n]} = (k_1, \dots, k_n)$, let functions $f_{k_i, \alpha_i} : I \rightarrow I, i \in \{1, \dots, n\}$ be defined by $f_{k_i, \alpha_i}(x) = k_i \cdot \ln(\alpha_i x + 1), x \in I$, where I is any of the considered intervals, see Example 5. Denote $\mathcal{F}_{[n]} = (f_{k_1, \alpha_1}, \dots, f_{k_n, \alpha_n})$. As we concluded in Example 5,

- (a) for I_1 , functions $f_{\frac{1}{\ln(\alpha_i+1)}, \alpha_i}$, $i \in \{1, \dots, n\}$, are concave and subadditive distortion functions,
- (b) in the cases of intervals I_2 and I_3 , for any $\alpha_i > 0$ and any $k_i > 0$, functions f_{k_i, α_i} are concave/ MU-concave and subadditive distortion functions.

Applying subadditive aggregation function $\max_{[n]}$ with distortion functions $\mathcal{F}_{[n]}$ in Formula (7), we obtain the aggregation function

$$\max_{[n]}^{\mathcal{F}_{[n]}}(a_1, \dots, a_n) = \max_{[n]} \left(f_{\frac{1}{\ln(\alpha_1+1)}, \alpha_1}(a_1), \dots, f_{\frac{1}{\ln(\alpha_n+1)}, \alpha_n}(a_n) \right) \quad (13)$$

for $(a_1, \dots, a_n) \in [0, 1]^n$, in the case of interval I_1 , and the aggregation function

$$\max_{[n]}^{\mathcal{F}_{[n]}}(a_1, \dots, a_n) = \max_{[n]} \left(f_{k_1, \alpha_1}(a_1), \dots, f_{k_n, \alpha_n}(a_n) \right) \quad (14)$$

for $(a_1, \dots, a_n) \in I^n$ and intervals I_2 and I_3 .

In all cases of intervals and parameters k_i and α_i , functions f_{k_i, α_i} are continuous/MU-continuous and concave/MU-concave. On the other hand, aggregation function $\max_{[n]}$ is continuous and subadditive, so that by Theorems 7 and 9. (a), aggregation function $\max_{[n]}^{\mathcal{F}_{[n]}}$ is continuous and subadditive. From the above considerations, it is easy to see that $\max_{[n]}^{\mathcal{F}_{[n]}}$ has properties (a12), (a13) and (a14). ✓

8. Conclusion

Distortion functions represent a suitable model for information transformation. Related to the nature and meaning of information, choosing distortion functions we can model desirable transformation of information before we include it in information fusion by aggregation function. Choosing sequence of distortion functions and aggregation function with appropriated properties, we obtain new aggregation function with some desirable properties. Various properties of constructed new aggregation function can be relevant for constructing new measure-like functions as distance functions, fuzzy measures, fuzzy metrics etc. For example, in [47] subadditive functions are used in the construction of generalized Choquet integral. Our future research will be focused on construction and investigation of distortion functions and aggregation functions suitable for construction of distance functions and fuzzy measures which are good models for certain decision making criterions, as for example for construction of distance functions for applications in image segmentation and other image processing tasks.

CRedit authorship contribution statement

Ljubo Nedović: Investigation, Conceptualization, Methodology. **Endre Pap:** Investigation, Conceptualization, Methodology. **Dorđe Dragić:** Investigation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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