

ON A PROPERTY OF THE ELLIPTIC BILINEAR FORM BY NONCONFORMING FINITE ELEMENTS

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Abstract

The paper deals with some nonconforming planar elements which fulfill an interesting and useful property. Namely, the elliptic bilinear form $a(u, v)$ of any function u coincides with its finite element interpolation $a(i_h u, v)$ for v belonging to the finite element space V_h . This property is proved for second order boundary value problems and it is applicable for some superconvergence FE algorithms. The common feature of considered cases is that the finite elements have integral type degrees of freedom.

Keywords: finite elements, interpolation operator, bilinear form, nonconforming methods

1. INTRODUCTION

Consider Poisson's equation

$$-\Delta u = f \text{ in } \Omega, \quad (1)$$

where Δ denotes the Laplace operator and Ω is bounded domain in R^2 . We also consider Dirichlet boundary conditions

$$u = 0 \text{ in } \partial\Omega, \quad (2)$$

where $\partial\Omega$ is Lipschitz continuous.

To formulate the variational equivalent of (1), (2), we define a variational space V :

$$V := H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}.$$

Here, $H^k(\Omega)$ is the usual Sobolev space of order k and $(\cdot, \cdot)$ denotes the inner product in $L_2(\Omega)$.

The variational problem associated with (1), (2) is given by:

$$a(u, v) = (f, v) \quad \forall v \in V, \quad (3)$$

where $a(u, v) = \iint_{\Omega} \nabla u \cdot \nabla v \, dx \, dy \quad \forall u, v \in V,$

which is continuous on $H^1(\Omega)$ and coercive on V .

We approximate (3) by nonconforming Finite Element Methods (FEMs). Let $\{\tau_h\}$ be a family of triangulations of Ω such that any

two triangles in τ_h share at most a vertex or an edge.

Let h be the mesh parameter, namely $h = \max_{K \in \tau_h} h_K$, with h_K being the diameter of the triangle/rectangle K . We suppose that the family of triangulations $\{\tau_h\}$ is quasi-uniform [1].

Let V_h be the nonconforming piecewise finite element spaces which we will precise. Then $V_h \subset L_2(\Omega)$ is the finite-dimensional collection of the admissible functions defined in four types finite elements, considered separately. Since V_h is not a subset of V we define the following bilinear form:

$$a_h(u, v) = \sum_{K \in \tau_h} \iint_K \nabla u \cdot \nabla v \, dx \, dy \quad \forall u, v \in V_h.$$

So, the nonconforming FE approximate problem is defined by:

$$a_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h. \quad (4)$$

2. NONCONFORMING FINITE ELEMENTS

We consider nonconforming finite element of following types:

- the integral-type Crouzeix-Raviart element (denoted C-R);

- the extension of C-R element (denoted EC-R);
- the rotated Q_1 – element (denoted Q_1^{rot});
- the extension of Q_1^{rot} (denoted EQ_1^{rot}).

These elements are depicted in Fig. 1 below.

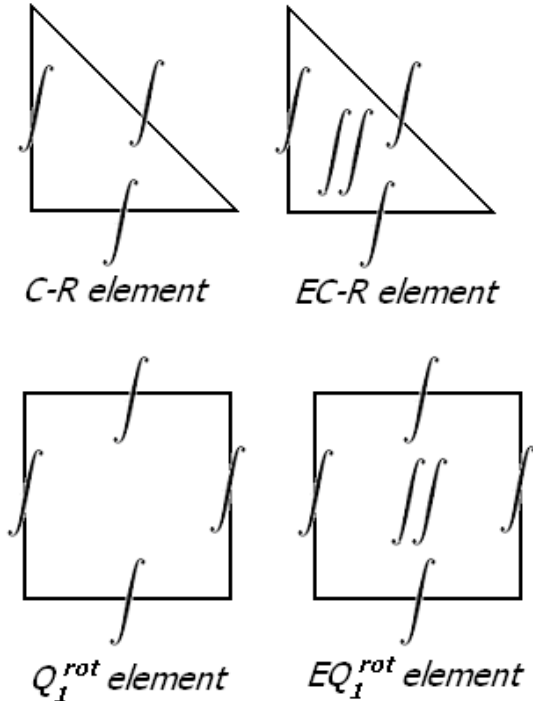


Fig. 1

I. Linear Crouzeix-Raviart (C-R) element of integral type

The piecewise interpolation functions $i_h u \in P_1 = \text{span}\{1, x, y\}$ are formulated as

$$\int_{l_j} i_h u \, dl = \int_{l_j} u \, dl, \quad j = 1, 2, 3,$$

where l_j are the edges of any triangle $K \in \tau_h$.

II. Extended Crouzeix-Raviart (EC-R) element of integral type

In this case $i_h u \in \text{span}\{1, x, y, x^2 + y^2\}$ (see [2]) are formulated under conditions

$$\begin{aligned} \int_{l_j} i_h u \, dl &= \int_{l_j} u \, dl, \quad j = 1, 2, 3, \\ \iint_K i_h u \, dx \, dy &= \iint_K u \, dx \, dy, \end{aligned}$$

and l_j are the edges of any triangle $K \in \tau_h$.

III. Rotated Q_1 – element (Q_1^{rot})

The piecewise interpolation functions $i_h u \in \text{span}\{1, x, y, x^2 - y^2\}$ are presented by

$$\int_{l_j} i_h u \, dl = \int_{l_j} u \, dl, \quad j = 1, 2, 3, 4,$$

where l_j are the edges of any rectangle $K \in \tau_h$.

IV. Extension of Q_1^{rot} element (EQ_1^{rot})

This time we take $i_h u \in \text{span}\{1, x, y, x^2, y^2\}$ and [3]:

$$\begin{aligned} \int_{l_j} i_h u \, dl &= \int_{l_j} u \, dl, \quad j = 1, 2, 3, 4, \\ \iint_K i_h u \, dx \, dy &= \iint_K u \, dx \, dy. \end{aligned}$$

And, again, here l_j are the edges of any rectangle $K \in \tau_h$.

3. MAIN RESULT

Let us consider the variational problem (4). We analyze a property of the form which is necessary for some postprocessing techniques from theoretical point of view. Moreover, all the elements presented in the previous section fulfill the conditions of integral type which are important to be satisfied in order to construct and implement a patch-recovery postprocedure accelerating the rate of convergence [4].

The next theorem contains the main result in this paper:

Theorem 1. Let τ_h consist of nonconforming elements K among those presented above by I-IV (Fig. 1).

If $u \in H_0^1(\Omega)$, then for any $v \in V_h$

$$a_h(i_h u, v) = a_h(u, v). \quad (6)$$

Proof. For $v \in V_h$, we have

$$\begin{aligned} a_h(i_h u - u, v) &= \sum_{K \in \tau_h} \iint_K \nabla(i_h u - u) \cdot \nabla v \, dx \, dy \\ &= \sum_{K \in \tau_h} \iint_K (\partial_x(i_h u - u) \partial_x v + \partial_y(i_h u - u) \partial_y v) \, dx \, dy. \end{aligned} \quad (7)$$

We adopt the following notations:

- When K is a triangle, the equations of the edges are respectively:

$$l_1 : x - x_0 + y - y_0 = 0; l_2 : x - x_0 = -h;$$

$$l_3 : y - y_0 = -h,$$

where (x_0, y_0) is the midpoint of the edge l_1 (Fig. 2a);

- In case of rectangular element the equations of the edges are respectively:

$$l_{1;3} : y - y_0 = \mp h; l_{2;4} : x - x_0 = \mp h,$$

where (x_0, y_0) is the center of K (Fig. 2b).

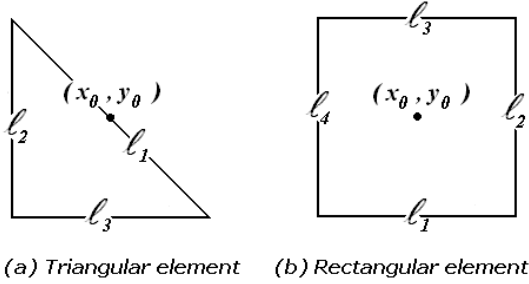


Fig. 2

I. C-R element

Having in mind that $\partial_x v = \text{const}$ and $\partial_y v = \text{const}$, we obtain:

$$\iint_K \partial_x (i_h u - u) \partial_x v \, dx \, dy$$

$$= \left(\int_{l_1} - \int_{l_2} \right) (i_h u - u) \partial_x v \, dy = 0.$$

By analogy, for the second term of the right-hand side of (7): $\iint_K \partial_y (i_h u - u) \partial_y v \, dx \, dy = 0$.

Thus, for any $v \in V_h$, $a_h(i_h u - u, v) = 0$.

II. EC-R element

Since K is an incomplete quadratic element, for $v \in V_h$ we can write:

$$v(x, y) = v(x_0, y_0) + (x - x_0) \partial_x v(x_0, y_0) + (y - y_0) \partial_y v(x_0, y_0) + \frac{1}{2} (x - x_0)^2 \partial_{xx} v(x_0, y_0) + \frac{1}{2} (y - y_0)^2 \partial_{yy} v(x_0, y_0). \quad (8)$$

Then,

$$\partial_x v(x, y) = \partial_x v(x_0, y_0) + (x - x_0) \partial_{xx} v,$$

$$\partial_y v(x, y) = \partial_y v(x_0, y_0) + (y - y_0) \partial_{yy} v,$$

and in addition, $\partial_{xx} v = \partial_{yy} v = \text{const}$.

Thus, we obtain:

$$\iint_K \partial_x (i_h u - u) \partial_x v \, dx \, dy$$

$$= \iint_K \partial_x (i_h u - u) \partial_x v(x_0, y_0) \, dx \, dy$$

$$+ \iint_K \partial_x (i_h u - u) (x - x_0) \partial_{xx} v \, dx \, dy$$

$$= \partial_x v(x_0, y_0) \left(\int_{l_1} - \int_{l_2} \right) (i_h u - u) \, dy$$

$$+ \partial_{xx} v \left(\int_{l_1} - \int_{l_2} \right) (i_h u - u) (x - x_0) \, dy$$

$$- \iint_K (i_h u - u) \partial_{xx} v \, dx \, dy.$$

The first and the third terms disappear and then

$$\iint_K \partial_x (i_h u - u) \partial_x v \, dx \, dy$$

$$= \partial_{xx} v \int_{l_1} (x - x_0) (i_h u - u) \, dy + h \partial_{xx} v \int_{l_2} (i_h u - u) \, dy$$

$$= \partial_{xx} v \int_{l_1} (x - x_0) (i_h u - u) \, dy.$$

By analogy,

$$\iint_K \partial_y (i_h u - u) \partial_y v \, dx \, dy$$

$$= \partial_{yy} v \int_{l_1} (y - y_0) (i_h u - u) \, dx.$$

Then, from $\partial_{xx} v = \partial_{yy} v$, we easily get $a_h(i_h u - u, v) = 0$.

III. Q_1^{rot} element

In this case, the presentation (8) is valid and besides

$$\partial_{xx} v = -\partial_{yy} v = \text{const}. \quad (9)$$

We calculate:

$$\begin{aligned}
& \iint_K \partial_x (i_h u - u) \partial_x v \, dx \, dy \\
&= \partial_x v(x_0, y_0) \left(\int_{l_2} \int_{l_4} (i_h u - u) \, dy \right) \quad (10) \\
&+ h \partial_{xx} v \left(\int_{l_2} \int_{l_4} (i_h u - u) \, dy \right) \\
&- \iint_K (i_h u - u) \partial_{xx} v \, dx \, dy = - \iint_K (i_h u - u) \partial_{xx} v \, dx \, dy.
\end{aligned}$$

Using the same arguments, it follows that

$$\begin{aligned}
& \iint_K \partial_y (i_h u - u) \partial_y v \, dx \, dy \\
&= - \iint_K (i_h u - u) \partial_{yy} v \, dx \, dy. \quad (11)
\end{aligned}$$

Finally, from (9) we obtain

$$\iint_K \nabla (i_h u - u) \cdot \nabla v \, dx \, dy = 0$$

and therefore $a_h(i_h u - u, v) = 0$.

IV. EQ_I^{rot} element

In this case relation (8) is fulfilled for any $v \in V_h$ as well as (10) and (11).

Then

$$\iint_K \nabla (i_h u - u) \cdot \nabla v \, dx \, dy = - \iint_K (i_h u - u) \Delta v \, dx \, dy.$$

Having in mind that $\Delta v = \text{const}$, the last integral is equal to zero because of the condition $\iint_K (i_h u - u) \, dx \, dy = 0$. Summarizing over all $K \in \tau_h$, we get $a_h(i_h u - u, v) = 0$.

Thus, the equality (6) is valid for all considered nonconforming finite elements.

Remark1. As a consequence from a Poincare type inequality for these classes of functions (see for example [5]) that

$$\|u - i_h u\|_{L_2(K)} \leq C h \|\nabla u - \nabla i_h u\|_{L_2(K)}.$$

4. CONCLUSION

The meaning of the equation (6) could be explained by the following way. It is well known [5] that in the convergence analysis of nonconforming FEM for the problem (3) we estimate

$$|a_h(i_h u - u, v)| \leq C |i_h u - u|_{1,h} |v|_{1,h},$$

where $|\cdot|_{1,h}$ is a first-order mesh-dependent seminorm. Thus, equality (6) could be explored for some superconvergence FE algorithms.

On the other hand, we do not claim the uniqueness of the presented nonconforming elements verifying (6). In this way, the problem discussed in this paper remains still open.

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