

HILL AND MATTHEW'S EQUATIONS AND SOME PROPERTIES OF THEIR PARTICULAR SOLUTIONS

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Abstract

In this work we have offered several theorems on oscillating and particular solutions of Hill's equation which we have not found in the well known monographs [1] and [2]. Namely, we have shown conditions under which Hill's equation is equation of oscillations. We have also shown that if the coefficient of the equation is periodic function with the period 2π , then Hill's equation also has periodic particular solution with the period 2π and one equation cannot have two fundamental particular solutions with the same period. By using the first Liouville's formula we have shown that the poles of the first order of the coefficient $\Phi(x)$ are simultaneously zeros of the solution of the equation, which is used for solving many Hill's equations. Namely, Hill's equation is most frequently reduced by substitution $y = e^{\int Z dx}$ to Riccati's equation, so it is very important to determine the form of its particular solution.

Key words: *Differential equations, periodicity and period, oscillatority, perturbation of periodicity, iterations.*

The equation of the following form:

$$y'' + \Phi(x) \cdot y = 0, \quad (1)$$

where $\Phi(x)$ is periodic equation, is Hill's equation which used to have and still has the important role in theory and practice of differential equations. According to the monograph by Magnus, Winkler, Hill's equation, New York, 1965, only until 1960 there had been registered some 300 mathematic works on Hill's equation and about some 700 works on the application of this equation in different sciences.

The main question which has been asked through the history of this equation, almost 120 years long is whether and when the equation (1) has periodic solutions, i.e. one class of periodic solutions or all periodic solutions, when we say that the solutions are in coexistence.

The condition necessary for differential equation to have periodic solutions if all coefficients of the equation are constants or periodic functions with the same period is fulfilled with Hill's equation. However, the

literature does not state clear enough if periodicity or non-periodicity of the integral $\int \Phi(x) dx$ is a sufficient condition for periodicity of one or both particular solutions of the equation.

The complicity of the problem of sufficient conditions is obviously seen from the first integral of the equation (1), because

$$y' = C_1 - \int_0^x \Phi(x) \cdot y(x) dx. \quad (2)$$

Therefore, after the partial integration is complete, we obtain:

$$y' = C_1 + y(x) \int_0^x \Phi(x) dx - \int_0^x \left(\int_0^x y'(x) \cdot \Phi(x) dx \right) dx \quad (3)$$

From the relation (3) it follows that the periodicity of the coefficient $\Phi(x)$ is definitely insufficient for the periodicity of the integral of the equation.

Some sufficient conditions for the periodicity of the solutions of Hill's equation, but only for some classes of coefficient $\Phi(x)$ have been found in the sequences of Ajns, Juetker, Campbell and others.

However, if the coefficient $\Phi(x)$ of the equations (1) is an interruptible function, then the function $\Phi(x) \cdot y$ is also interruptible. Therefore, the standard theorem on existence of solutions cannot be applied for the Hill's equation. Let x_0 be the pole of the first order of the coefficient $\Phi(x)$. Then:

$$\Phi(x) = \frac{a_0}{x-x_0} + a_1 + a_2(x-x_0) + a_3(x-x_0)^2 + \dots \quad (4)$$

If we suppose that the equation (1) has analytic solution

$$y = C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + \dots \quad (5)$$

we obtain that

$$y'' = 2 \cdot 1 \cdot C_2 + 3 \cdot 2 \cdot C_3(x-x_0) + 4 \cdot 3 \cdot C_4(x-x_0)^2 + \dots$$

Therefore, from (1), it follows:

$$\begin{aligned} \Phi(x) &= \frac{2 \cdot 1 \cdot C_2 + 3 \cdot 2 \cdot C_3(x-x_0) + 4 \cdot 3 \cdot C_4(x-x_0)^2 + \dots}{C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + C_3(x-x_0)^3 + C_4(x-x_0)^4} = \\ &= \frac{a_0}{x-x_0} + a_1 + a_2(x-x_0) + a_3(x-x_0)^2 + a_4(x-x_0)^3 \end{aligned}$$

Further, there follows the identity:

$$\begin{aligned} &2 \cdot 1 \cdot C_2(x-x_0) + 3 \cdot 2 \cdot C_3(x-x_0)^2 + \\ &4 \cdot 3 \cdot C_4(x-x_0)^3 + \dots = \\ &\equiv \frac{a_0 C_0}{x-x_0} + (a_1 C_0 + a_0 C_1) + (a_2 C_0 + a_1 C_1 + a_0 C_2) \cdot (x-x_0) + \\ &(a_3 C_0 + a_2 C_1 + a_1 C_2 + a_0 C_3) \cdot (x-x_0)^2 + \dots, \end{aligned}$$

so for the required analytic solution, the following is valid:

$$\begin{aligned} a_0 C_0 &= 0, \\ a_1 C_0 + a_0 C_1 &= 2C_2 \end{aligned}$$

$$\begin{aligned} a_2 C_0 + a_1 C_1 + a_0 C_2 &= 3 \cdot 2 \cdot C_3 \\ a_3 C_0 + a_2 C_1 + a_1 C_2 + a_0 C_3 &= 4 \cdot 3 \cdot C_4 \end{aligned}$$

and so on, from which, since $a_0 \neq 0$, it follows that $C_0 = 0$, i.e.

$$y(x) = (x-x_0) \cdot [C_1 + C_2(x-x_0) + C_3(x-x_0)^2 + \dots], \quad (6)$$

which means that the pole x_0 of the coefficient $\Phi(x)$ is zero of the first order of the solution of the Hill's equation.

Theorem 1. *The poles of the first order of the coefficient $\Phi(x)$ are simultaneously zeros of the first order of particular solution of the Hill's equation and vice versa, i.e. the zeros of the first order of the particular integral of Hill's equation are poles of the first order of the coefficient $\Phi(x)$.*

By applying this theorem, a great number of Hill's equations can be solved.

However, if zero solutions and poles of the coefficient of Hill's equation are not connected in the way specified by the theorem 1, then it is possible to determine the solution of Hill's equation. Nevertheless, that solution is not analytic. In that case, a non-characteristic integral and its main value should be observed.

$$v \cdot p \int_{x_0-\varepsilon}^{x_0+\varepsilon} \Phi(x) \cdot y(x) dx$$

All this shows the difficulties which occur during the integration of Hill's equation and why the sufficient conditions for the existence of its periodic solutions have not been determined up to today.

In the work [1], we have shown that, if the equation (1) has periodic solutions, the integral $\int \Phi(x) dx$ must be non-periodic and that is why Hill's equation is usually written in the following form:

$$y'' + [\lambda + \Phi(x)] \cdot y = 0 \quad (7)$$

From this, there follows that the integral of the coefficient of the equation is not periodic function, because it contains the linear part $\lambda \cdot x$.

We have given a modest contribution to the theory of Hill's equation, by adding some details, but only for the case when the

coefficient of the equation is continuous positive function

Hill's equation (7) has trigonometric (oscillating) solutions in the form of iteration sequences:

$$\begin{cases} y_1 = \cos_{\Phi} x = 1 - \int \int (\Phi + \lambda) + \int \int (\Phi + \lambda) \int \int (\Phi + \lambda) - \dots \\ y_2 = \sin_{\Phi} x = x - \int \int (\Phi + \lambda) + \int \int (\Phi + \lambda) \int \int (\Phi + \lambda) - \dots \end{cases} \quad (8)$$

if $\Phi(x) + \lambda$ is positive continuous and sufficiently big function, which can cause oscillations.

Theorem 2. If y_1 and y_2 are particular solutions determined with iterations, then the equation (7) has general solution

$$y(x) = I_{\Phi} + I_{\lambda} + I_{\Phi, \lambda} \quad (9)$$

where I are iterations, so that:

- I_{Φ} are iterations which depend only on $\Phi(x)$
- I_{λ} are iterations which depend only on λ
- $I_{\Phi, \lambda}$ are iterations which depend both on $\Phi(x)$ and λ

The proof to this theorem is trivial, because the functional sequences which we have obtained by iterations for $\cos_{\Phi, \lambda} x$ and $\sin_{\Phi, \lambda} x$ can be written in the following form:

$$\begin{aligned} \cos_{\Phi, \lambda} x = & [1 - \int \int \Phi + \int \int \Phi \int \int \Phi - \int \int \Phi \int \int \Phi \int \int \Phi + \dots] + \\ & + [\int \int \lambda \int \int \Phi + \int \int \Phi \int \int \lambda + \dots] + [- \int \int \lambda + \int \int \lambda \int \int \lambda - \int \int \lambda \int \int \lambda \int \int \lambda + \dots] \end{aligned}$$

and analogically

$$\begin{aligned} \sin_{\Phi, \lambda} x = & [x - \int \int x \Phi + \int \int \Phi \int \int x \Phi - \int \int \Phi \int \int \Phi \int \int x \Phi + \dots] + \\ & + [\int \int \lambda \int \int x \Phi + \int \int \Phi \int \int x \lambda + \dots] + [\int \int x \lambda + \int \int \lambda \int \int x - \int \int \lambda \int \int \lambda \int \int x \lambda + \dots] \end{aligned}$$

Theorem 3. If $\Phi(x)$ is real periodic function with the period $\omega = 2\pi$, then the equation (7) has one periodic solution also with the period 2π .

Proof. Let $\Phi(x)$ be periodic function with the period 2π , i.e. let us have:

$$\Phi(x + 2\pi) = \Phi(x)$$

From the equation (1), it follows that:

$$\frac{y''(x)}{y(x)} = -[\lambda + \Phi(x)] = -[\Phi(x + 2\pi) + \lambda],$$

which means that the quotient $\frac{y''(x)}{y(x)}$ is periodic function with the period 2π , the following is valid:

$$\frac{y''(x + 2\pi)}{y(x + 2\pi)} = \frac{y''(x)}{y(x)} = -[\Phi(x) + \lambda],$$

i.e.

$$y''(x + 2\pi) + [\Phi(x) + \lambda] \cdot y(x + 2\pi) = 0,$$

which means that there is one periodic integral of the Hill's equation with the period 2π .

Theorem 4. Hill's equation (7) cannot have two fundamental particular solutions with the period 2π .

Proof. Let y_1 be particular solution of Hill's equation with the period 2π . In that case there is:

$$y_1(x + 2\pi) = y_1(x), y_1'(x + 2\pi) = y_1'(x), y_1''(x + 2\pi) = y_1''(x),$$

because the derivative of any order of the periodic function is also periodic function with the same period. If we introduce new unknown function Z into equation (1), by making substitution:

$$y = y_1 \cdot Z, \quad Z = Z(x)$$

by differentiating we obtain:

$$\begin{aligned} y' &= y_1' \cdot Z + y_1 \cdot Z' \\ y_1'' &= y_1'' \cdot Z + 2y_1' \cdot Z' + y_1 \cdot Z'' \end{aligned}$$

By making substitution y and y'' in the equation (7), we obtain:

$$y_1'' \cdot Z + 2y_1' \cdot Z' + \{y_1'' + [\lambda + \Phi(x)] \cdot y_1\} \cdot Z = 0.$$

From the last equation, considering that the integral y_1 identically fulfills the equation (7), it follows that:

$$y_1 \cdot Z'' + 2y_1' \cdot Z' = 0$$

After dividing with $y_1 \cdot Z'$, we obtain the following equation:

$$\frac{Z''}{Z'} + 2 \frac{y_1'}{y_1} = 0,$$

or the equation

$$\frac{dZ'}{Z'} + 2 \frac{dy_1}{y_1} = 0 \quad (10)$$

The equation (10) is equation with the separated variable, so by quadratures we obtain:

$$\ln Z + \ln y_1^2 = \ln C_1$$

i.e.

$$Z' = \frac{C_1}{y_1^2}$$

Finally, we obtain:

$$Z = C_1 \int_0^x \frac{dx}{y_1^2}$$

If we go back to the old variable, then there is:

$$y = C_1 \cdot y_1 \cdot \int_0^x \frac{dx}{y_1^2}$$

For $C_1 = 1$ we obtain particular integral:

$$y_2 = y_1 \cdot \int_0^x \frac{dx}{y_1^2} \quad (11)$$

The equation (11) is a first known Liouville's formula which gives connection between particular integrals of Hill's equation. Let us now prove that the integral (11) cannot be periodic function with the period 2π .

Since

$$y_2(x+2\pi) = y_1(x+2\pi) \int_0^{x+2\pi} \frac{dx}{y_1^2} = y_1(x) \left(\int_0^x \frac{dx}{y_1^2} + \int_x^{x+2\pi} \frac{dx}{y_1^2} \right) =$$

$$= y_1(x) \int_0^x \frac{dx}{y_1^2} + y_1(x) \int_x^{x+2\pi} \frac{dx}{y_1^2} = y_2(x) + y_1(x) \int_x^{x+2\pi} \frac{dx}{y_1^2} \neq y_2(x)$$

then $y_2(x)$ is not periodic integral with the period 2π . The solution (11) in the point

$x+2\pi$ can also be written in the following form:

$$\begin{aligned} y_2(x+2\pi) &= y_1(x) \left[\int_0^x \frac{d(x+2\pi)}{y_1^2(x+2\pi)} + \int_x^{x+2\pi} \frac{dx}{y_1^2} \right] = \\ &= y_1(x) \int_0^x \frac{dx}{y_1^2} + y_1(x) \int_x^{x+2\pi} \frac{dx}{y_1^2} = \\ &= y_2(x) + \theta \cdot y_2(x) \end{aligned}$$

where $0 < \theta < 1$.

Therefore, if the function $\Phi(x)$ has a period 2π , the equation (7) cannot have two linearly independent solutions with the period 2π , which is in accordance with Floke's theorem. However, as a rule, the equation has solution $y(x)$, which in case of the right choice of the real or complex number μ that is called characteristic exponent, fulfills the functional equation.

$$y(x+2\pi) = e^{2\pi\mu} \cdot y(x)$$

The solution $y(x)$ is periodic or semi-periodic depending on the fact if $e^{2\pi\mu}$ equals +1 or -1. The appropriate values of λ and $\bar{\lambda}$ parameters are called periodic or semi-periodic characteristic values.

Differential Matthew's equation is a special case of the Hill's equation.

$$y'' + (a \cdot \cos 2x + b) \cdot y = 0 \quad (12)$$

This equation introduces very important Matthew's functions into technique. Although the coefficient of the equation (12) $\Phi(x) = a \cdot \cos 2x + b$ is periodic, while the integral $\int \Phi(x) dx = \int (a \cdot \cos 2x + b) dx =$

$$= bx + \frac{a}{2} \sin 2x \text{ is non-periodic, the equation}$$

has periodic solutions. However, Matthew's equation as a canonic equation of the second order for $b > a > 0$ fulfills the conditions of the classic theorem on oscillations, because:

$$1^\circ \quad \Phi(x) = b + a \cdot \cos 2x > 0 \text{ for}$$

$$x \in (0, +\infty) \text{ and } b > a > 0$$

$$2^\circ \quad \int_0^{+\infty} \Phi(x) dx = \int_0^{+\infty} (b \cdot \cos 2x + a) dx = +\infty$$

Thus the equation (12) has oscillating solutions in the form of iteration sequences:

$$y_1 = 1 - \int_0^x dx \int_0^x (b + a \cdot \cos 2x) dx + \int_0^x dx \int_0^x (b + a \cdot \cos 2x) dx \\ \int_0^x dx \int_0^x (b + a \cdot \cos 2x) dx - \dots, \\ y_2 = x - \int_0^x dx \int_0^x x(b + a \cdot \cos 2x) dx + \int_0^x dx \int_0^x (b + a \cdot \cos 2x) dx \\ \int_0^x dx \int_0^x x(b + a \cdot \cos 2x) dx - \dots$$

Zero oscillations of y_1 and y_2 are cross sections of the curve $F(x) = x \cdot \sqrt{b + a \cdot \cos 2x}$ with horizontal lines $y = n\pi$ and $y = (n - \frac{1}{2})\pi$, therefore in the solutions of the equations:

$$x \cdot \sqrt{b + a \cos 2x} = n \cdot \pi, n = 0, 1, 2, \dots, \\ x \cdot \sqrt{b + a \cos 2x} = (2n - 1) \cdot \frac{\pi}{2}, n = 1, 2, 3, \dots,$$

Therefore, Hill's equation (7) and equations which are its special cases can have periodic solutions, if periodic coefficient $\Phi(x)$ changes the sign. If $y_1(x)$ is periodic solution of the equation (7), then, based on the Liouville's formula, the other particular integral has the following form:

$$y_2 = y_1 \int_0^x \frac{dx}{y_1^2}$$

and it can be periodic, but not necessarily. If y_1 does not change the sign, then $\frac{1}{y_1^2}$ is a complete periodic function which does not have zeros, while the integral $\int_0^x \frac{dx}{y_1^2}$ is not periodic, but it represents monotonously rising function as the integral of the positive function. Since solutions of the linear homogenous differential equation of the

second order can have only plain zeros, then $\frac{1}{y_1^2}$ has the poles of the second order and the

integral $\int_0^x \frac{dx}{y_1^2}$ is not characteristic. Since the poles of the second order of the integral $\int_0^x \frac{dx}{(x - x_0)^2} = \frac{1}{x_0 - x}$ are reversed into the poles of the first order, which are zeros x_0 of the solution y_1 , then there occurs the shortening in the integral y_2 and only non-zero part remains. This part can be periodic, but not necessarily.

Example. Hill's equation $y'' = \frac{2}{\cos^2 x} \cdot y$, which has positive, but interruptible coefficient $\Phi(x) = \frac{2}{\cos^2 x}$, has one interruptible periodic solution $y_1 = \operatorname{tg} x$, while the other particular integral

$$y_2 = y_1 \cdot \int_0^x \frac{dx}{y_1^2} = \operatorname{tg} x \cdot \int_0^x \frac{dx}{\operatorname{tg}^2 x} = \operatorname{tg} x \cdot (x \cdot \operatorname{tg} x + 1),$$

is interruptible, but not the periodic function. However, the general solution of this equation $y = C_1 \operatorname{tg} x + C_2 \operatorname{tg} x \cdot (1 + x \cdot \operatorname{tg} x)$ has the sub-class of periodic solutions and sub-class of non-periodic, but oscillating solutions.

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