

STURM'S THEOREMS FOR DIFFERENTIAL EQUATIONS WITH SINGULARITIES LEGENDRE'S EQUATION

Stana Cvejic

*Faculty of Natural Sciences and Mathematics
University in Kosovska Mitrovica, Serbia*

Milena Lekic

*Faculty of Natural Sciences and Mathematics
University in Kosovska Mitrovica, Serbia*

Abstract

Literature contains no special theorems on differential equations with singularities. It contains no theorems of the Sturm's type which would apply to canonic interval between singular points, either. And finally, there are no theorems on existence of the solutions of differential equation close to each singularity. Whether equation with singularity has oscillating solutions, primarily depends on the sign of the coefficient of the equation, monotony or non-monotony of the coefficient in the interval between two singularities, type of the singularity, as well as on the width of the interval between singularities. In this work we have tried and hopefully succeeded in determining conditions under which Legendre's equation $(x^2 - 1) \cdot y'' + xy' \pm n^2 y = 0$, which is differential equation with singularity, has oscillating solutions. In this way we have formed some premises on zeros of oscillations.

Key words: Differential equations, Legendre's equation, Sturm's theorems, special functions, singular points, types of singularity, oscillating solutions.

The equation of the following form

$$(x^2 - 1) \cdot y'' + xy' + n^2 y = 0 \quad (1)$$

is Legendre's equation, which is well known and historically very important differential equation both for techniques of application and theory of special functions. By introducing new, independent variable t , and making substitution:

$$x = cht, \quad y = y(t) \quad (2)$$

it becomes

$$(ch^2 t - 1) \cdot \frac{y'' \cdot sht - y'cht}{sh^3 t} + cht \cdot \frac{y'}{sht} + n^2 y = 0$$

or

$$sh^2 t \cdot \frac{y'' \cdot sht - y'cht}{sh^3 t} + \frac{cht}{sht} \cdot y' + n^2 y = 0,$$

i.e.

$$y'' - \frac{cht}{sht} \cdot y' + \frac{cht}{sht} \cdot y' + n^2 y = 0$$

Based on this we obtain the canonic equation:

$$y''(t) + n^2 \cdot y(t) = 0 \quad (3)$$

in which $y = y(t)$ is unknown function

The equation (3) is equation of harmonic oscillations and its general solution is:

$$y(t) = C_1 \cdot \cos nt + C_2 \cdot \sin nt, \quad (4)$$

where t is inverse function of the function (2),

i.e. of the function $x = cht = \frac{e^t + e^{-t}}{2}$, so there

is also $e^t = x \pm \sqrt{x^2 - 1}$ and finally:

$$t = \ln(x \pm \sqrt{x^2 - 1}) = A \cdot r \cdot chx, \quad x > 1. \quad (5)$$

Therefore, the exact solution of the equation (1) is

$$y = C_1 \cos(n \cdot r \cdot A \cdot chx) + C_2 \sin(n \cdot r \cdot A \cdot chx) = (6)$$

$$C_1 \cos(n(\ln(x \pm \sqrt{x^2 - 1}))) + C_2 \sin(n(\ln(x \pm \sqrt{x^2 - 1})))$$

which is the oscillating one.

Zeros of oscillations are solutions of the equations:

$$n \cdot r \cdot A \cdot chx = k\pi, k=0,1,2,\dots,$$

or of the equations:

$$A \cdot r \cdot chx = \frac{k\pi}{n}, \text{ or}$$

$$\ln(x \pm \sqrt{x^2 - 1}) = \frac{k\pi}{n}, k=0,1,2,\dots,$$

Therefore, it follows that

$$x \pm \sqrt{x^2 - 1} = e^{\frac{k\pi}{n}}$$

And finally there is:

$$x_{(k,n)} = ch \frac{k\pi}{n} = \frac{e^{\frac{k\pi}{n}} + e^{-\frac{k\pi}{n}}}{2} \quad (7)$$

Based on the relation (7) for the fixed n , zeros of oscillations marshal with the exponential speed with the increase of k . In order to check if something similar or at least approximate can be obtained with our methods, the equation (1) is reduced to canonic form, by introducing new, unknown function $Z=Z(x)$ and making substitution:

$$y = e^{-\frac{1}{2} \int \frac{xdx}{x^2-1}} \cdot Z = e^{-\frac{1}{2} \int \frac{d(x^2-1)}{x^2-1}} \cdot Z = e^{-\frac{1}{4} \ln(x^2-1)} \cdot Z,$$

so

$$y = (x^2 - 1)^{-\frac{1}{4}} \cdot Z, x > 1. \quad (8)$$

By differentiating, from (8) we obtain:

$$y' = -\frac{1}{4}(x^2 - 1)^{-\frac{5}{4}} \cdot 2xZ + (x^2 - 1)^{-\frac{1}{4}} \cdot Z'$$

$$y'' = \frac{5}{4}x^2 Z (x^2 - 1)^{-\frac{9}{4}} - \frac{1}{2} \cdot Z \cdot (x^2 - 1)^{-\frac{5}{4}} -$$

$$\frac{1}{2}xZ'(x^2 - 1)^{-\frac{5}{4}} - \frac{1}{2}xZ'(x^2 - 1)^{-\frac{5}{4}} + Z''(x^2 - 1)^{-\frac{1}{4}}$$

If we substitute y , y' and y'' in the equation (1) which we have written in the following form:

$$y'' + \frac{x}{x^2 - 1} \cdot y' + \frac{n^2}{x^2 - 1} \cdot y = 0,$$

we obtain:

$$(x^2 - 1)^{-\frac{1}{4}} \cdot Z + \left[-\frac{1}{2}(x^2 - 1)^{-\frac{5}{4}} x - \frac{1}{2}(x^2 - 1)^{-\frac{5}{4}} \cdot x + (x^2 - 1)^{-\frac{5}{4}} \cdot x \right] \cdot Z + \left[\frac{5}{4}x^2(x^2 - 1)^{-\frac{9}{4}} - \frac{1}{2}(x^2 - 1)^{-\frac{5}{4}} - \frac{1}{2}x^2(x^2 - 1)^{-\frac{9}{4}} + n^2(x^2 - 1)^{-\frac{5}{4}} \right] \cdot Z = 0$$

By multiplying with $(x^2 - 1)^{\frac{1}{4}}$, we obtain:

$$Z'' + \left[\left(\frac{5}{4} - \frac{1}{2} \right) x^2 - (x^2 - 1)^{-2} + (n^2 - \frac{1}{2}) \cdot (x^2 - 1)^{-1} \right] \cdot Z = 0,$$

Based on this, there follows the canonic equation:

$$Z'' + \frac{(n^2 + \frac{1}{4})x^2 - (n^2 - \frac{1}{2})}{(x^2 - 1)^2} \cdot Z = 0 \quad (9)$$

Coefficient $B(x)$ of canonic equation (9) is

$$\text{positive for every } x \text{ for which } x^2 > \frac{n^2 - \frac{1}{2}}{n^2 + \frac{1}{4}},$$

which is fulfilled for $n > \frac{1}{\sqrt{2}}$. However, the

second condition according to which $B(x)$ is at least of the following range $\frac{1}{\sqrt{x^\alpha}}$, $0 < \alpha < 2$

is not fulfilled, because $B(x) \sim \frac{n^2 x^2}{x^4} = \frac{n^2}{x^2}$.

Consequently, $\alpha = 2$, while there are sure oscillations only for $\alpha < 2$. Therefore, we cannot conclude that the equation (1) has oscillations of the $\sin_{B(x)} x$ and $\cos_{B(x)} x$ types. We also cannot determine zeros of the oscillating solutions of the equation (1) by applying our procedure, i.e. geometrically.

1. ANOTHER TYPE OF OSCILLATIONS IN SOLUTIONS OF LEGENDRE'S EQUATION

Let us have differential equation:

$$(x^2 - 1) \cdot y'' + xy' - n^2 y = 0, \quad (10)$$

which by introducing the familiar substitution:

$$x = \cos t, \quad |x| < 1 \quad (11)$$

can be solved elementarily.

Indeed, there is:

$$\sin^2 t \cdot \frac{y' \cdot \cos t - y'' \sin t}{\sin^3 t} - \cos t \cdot \frac{y'}{\sin t} - n^2 \cdot y = 0$$

There follows:

$$y'' - \frac{\cos t}{\sin t} \cdot y' + \frac{\cos t}{\sin t} \cdot y' - n^2 y = 0 \text{ and } y''(t) + n^2 \cdot y(t) = 0. \quad (12)$$

The solution of the equation (12) are harmonic oscillations:

$$y(t) = C_1 \cos(n \cdot t) + C_2 \sin(n \cdot t).$$

Since $t = \arccos x$, we obtain the exact solution:

$$y = C_1 \cos(n \arccos x) + C_2 \sin(n \arccos x). \quad (13)$$

The solution (13) is obviously oscillating and zeros of oscillations are in the solutions of the equations:

$$\begin{cases} n \cdot \arccos x = k\pi, k = 0, 1, 2, \dots, \\ n \cdot \arccos x = (k - \frac{1}{2})\pi, k = 1, 2, 3, \dots \end{cases}$$

From the above, we obtain:

$$x = \cos \frac{k\pi}{n}, \quad k=0, 1, 2, \dots, \quad (14)$$

$$x = \cos \frac{\left(k - \frac{1}{2}\right)\pi}{n}, \quad k=1, 2, 3 \dots \quad (15)$$

In the equation (11), number n can be natural, but not necessarily. If n is not natural number, then numbers k and n do not have to be co-measurable. In this case, solutions with zeros (14) and (15) represent random harmonic oscillations without joint period. However, if n is natural number, then, since $2k-1$ is also a natural number, the zeros (14) and (15) are co-measurable, while harmonic oscillations (13) are periodic. In this case the well known Chebyshev polynomials are obtained.

$$T_n(x) = 2^{-n+1} \cdot \cos(n \cdot \arccos x) \quad (16)$$

And now here we have the same problem. Can zeros (14) and (15) be obtained by applying our methods, at least approximately, which would lead to many other generalizations? That is why we reduce the equation (1), i.e. the equation

$$y'' + \frac{x}{x^2 - 1} \cdot y' - \frac{n^2}{x^2 - 1} \cdot y = 0$$

to canonic form. For that purpose we introduce the substitution:

$$y = e^{\frac{1}{2} \int \frac{x dx}{x^2 - 1}} \cdot Z = e^{\frac{1}{4} \int \frac{2x}{x^2 - 1} dx} \cdot Z = e^{\frac{1}{4} \ln(x^2 - 1)} \cdot Z = (x^2 - 1)^{\frac{1}{4}} \cdot Z$$

and obtain the same calculation as with the first type of Legendre's equation:

$$(x^2 - 1)^{\frac{1}{4}} \cdot Z'' + \left[\frac{5x^2}{4(x^2 - 1)^{\frac{9}{4}}} - \frac{x^2}{(x^2 - 1)^{\frac{9}{4}}} - \frac{1}{2(x^2 - 1)^{\frac{5}{4}}} - \frac{n^2}{2(x^2 - 1)^{\frac{5}{4}}} \right] \cdot Z = 0$$

or

$$Z'' + \left[\frac{\frac{3}{4}x^2}{(x^2 - 1)^2} - \frac{n^2 + \frac{1}{2}}{x^2 - 1} \right] \cdot Z = 0 \quad (17)$$

The equation (17) is oscillating if:

$$B(x) = \frac{\frac{3}{4}x^2 - (n^2 + \frac{1}{2}) \cdot (x^2 - 1)}{(x^2 - 1)^2} > 0,$$

i.e.

$$n^2 + \frac{1}{2} + x^2 \left(\frac{3}{4} - n^2 - \frac{1}{2} \right) > 0,$$

$$x^2 \left(\frac{1}{4} - n^2 \right) + n^2 + \frac{1}{2} > 0,$$

$$x^2 \left(n^2 - \frac{1}{4} \right) < n^2 + \frac{1}{2},$$

$$x^2 < \frac{n^2 + \frac{1}{2}}{n^2 - \frac{1}{4}}$$

where n is a natural number. Since, according to the introduced substitution (2), $|x| < 1$ and since the last inequality has already been fulfilled for $n=1$, then the equation (17) on a definite interval can have oscillating solutions. Let us form now the function of frequency:

$$F(x) = x \cdot B(x) = x \cdot \sqrt{\frac{\left(\frac{3}{4} - n^2 - \frac{1}{2}\right)x^2 + n^2 + \frac{1}{2}}{(x^2 - 1)^2}} \quad (18)$$

and test its cross section with the horizontals $y = k\pi$, i.e. let us determine solutions of the equation:

$$\frac{x}{|x^2 - 1|} \cdot \sqrt{\left(\frac{1}{4} - n^2\right)x^2 + n^2 + \frac{1}{2}} = k\pi, \quad (19)$$

i.e. of the following equation:

$$x \cdot \sqrt{\left(\frac{1}{4} - n^2\right)x^2 + n^2 + \frac{1}{2}} = k\pi \cdot |x^2 - 1|,$$

or of the following equation:

$$x^2 \cdot \left[\left(\frac{1}{4} - n^2\right)x^2 + n^2 + \frac{1}{2} \right] = k^2 \pi^2 (x^2 - 1)^2$$

By making substitution $x^2 = t$, the last equation is reduced to square equation

$$t \cdot \left[\left(\frac{1}{4} - n^2\right)t + n^2 + \frac{1}{2} \right] = k^2 \pi^2 (t - 1)^2$$

or to the equation:

$$\left[\left(\frac{1}{4} - n^2\right) - k^2 \pi^2 \right] \cdot t^2 + \left[\left(n^2 + \frac{1}{2}\right) + 2k^2 \pi^2 \right] \cdot t - k^2 \pi^2 = 0 \quad (20)$$

Now we face the question if the equation (20) has always a solution.

The curve (18), which we observe on the interval $[0,1)$, monotonously rise from zero toward $+\infty$, while the straight line $x=1$ is its vertical asymptote from the left. For $k=0$, the cross section of the curve $F(x)$ and horizontal is located in the point $x=0$, while for every other k , there is a cross section ξ_k , $0 < \xi_k < 1$ of the function of frequency $F(x)$ and horizontal line $y = k\pi$.

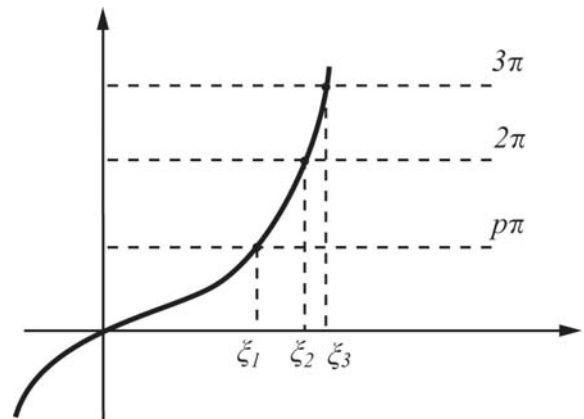


Fig. 1

This means that square equation (20) has solutions and those solution are t_1 and t_2 where

$$t_{1/2} = \frac{-\left[\left(n^2 + \frac{1}{2}\right) + 2k^2 \pi^2\right] \pm \sqrt{\left[\left(n^2 + \frac{1}{2}\right) + 2k^2 \pi^2\right]^2 + 4k^2 \pi^2 \left[\left(\frac{1}{4} - n^2\right) - k^2 \pi^2\right]}}{2\left[\left(\frac{1}{4} - n^2\right) - k^2 \pi^2\right]}$$

Therefore, zeros of oscillations are:

$$\xi_k = \sqrt{\frac{-\left[\left(n^2 + \frac{1}{2}\right) + 2k^2\pi^2\right] \pm \sqrt{\left[\left(n^2 + \frac{1}{2}\right) + 2k^2\pi^2\right]^2 + 4k^2\pi^2\left[\left(\frac{1}{4} - n^2\right) - k^2\pi^2\right]}}{2\left[\left(\frac{1}{4} - n^2\right) - k^2\pi^2\right]}} \quad (21)$$

where n is a fixed number, while $k=0,1,2,\dots$

Therefore, we have obtained sequence of zeros of oscillations ξ_k which monotonously tend toward one and which are actually approximate zeros (14) obtained by exact solving.

CONCLUSION

It is very complicated to find out whether the equation with the singularities has oscillating solutions, because such equations can be solved by iterations only on intervals between singularities, by isolation of the singular point, when it is possible. In order to apply our theorems, the equation should be reduced to canonic form and check if the coefficient of the canonic equation is positive and big enough to cause oscillations. It actually means that on canonic interval the function of the frequency $F(x)$ has cross sections with horizontals $y = k \cdot \pi$ and

$y = (2k-1) \cdot \frac{\pi}{2}$. It also means that on a

indefinite interval the integral $\int_0^{+\infty} B(x)dx$

diverges or that the coefficient $B(x)$ monotonously tends toward zero i .e. it is in the range $B(x) = \frac{1}{x^\alpha}$, $0 < \alpha < 2$. The canonic equation of the Legendre's equation

$(x^2 - 1) \cdot y'' + xy' + n^2 y = 0$ has coefficient $B(x)$ in the range $\frac{1}{x^\alpha}$, $\alpha = 2$, so we cannot determine if it has oscillations of the $\sin_{B(x)} x$ and $\cos_{B(x)} x$ type. However, for the equation $(x^2 - 1) \cdot y'' + xy' - n^2 y = 0$ we have shown that it has oscillating solutions in the interval $[0,1)$ and that zeros of the oscillations monotonously tend toward one.

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